Time evolution of the wave equation using the Rapid Expansion Method for highly accurate RTM

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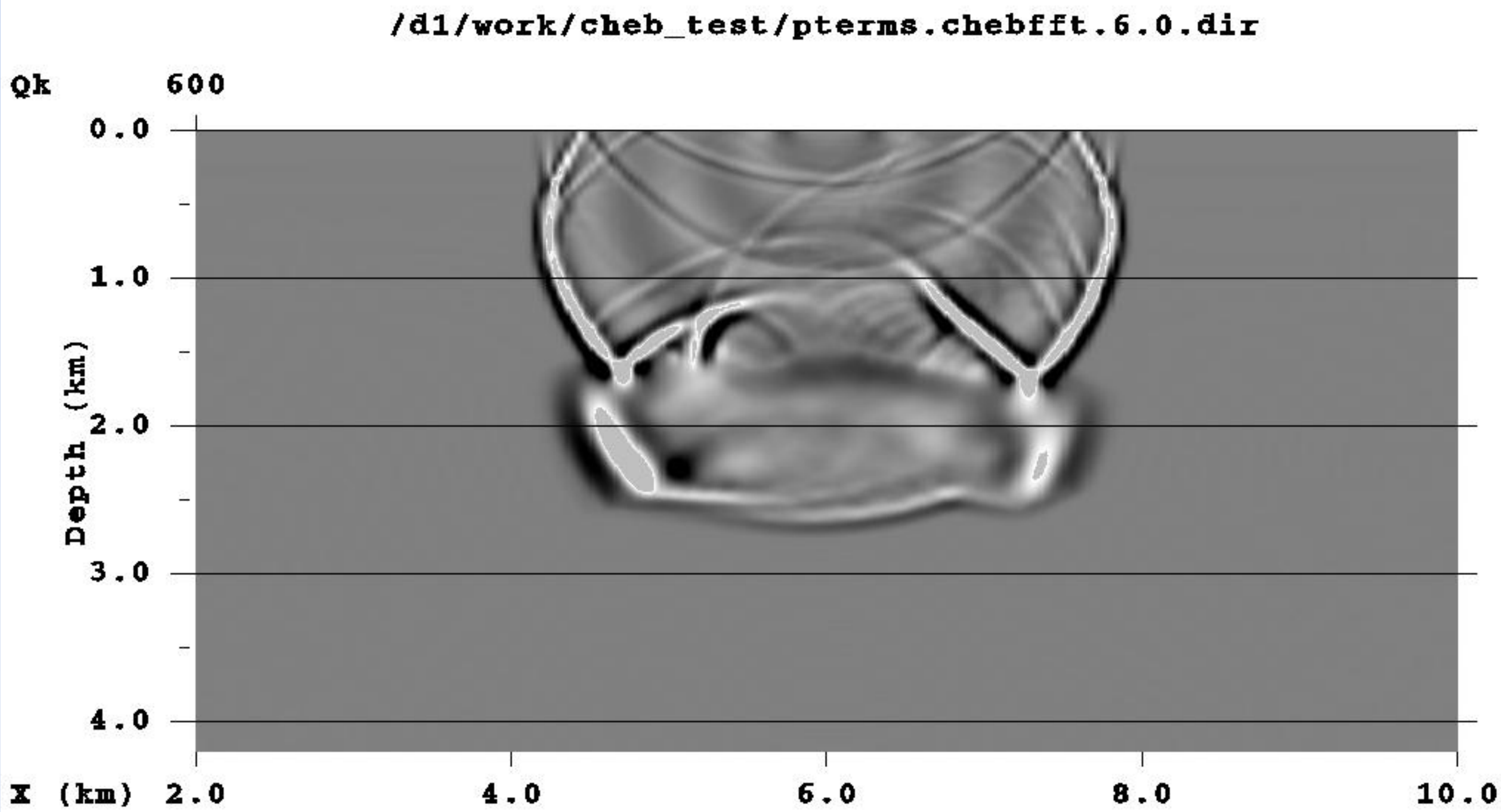
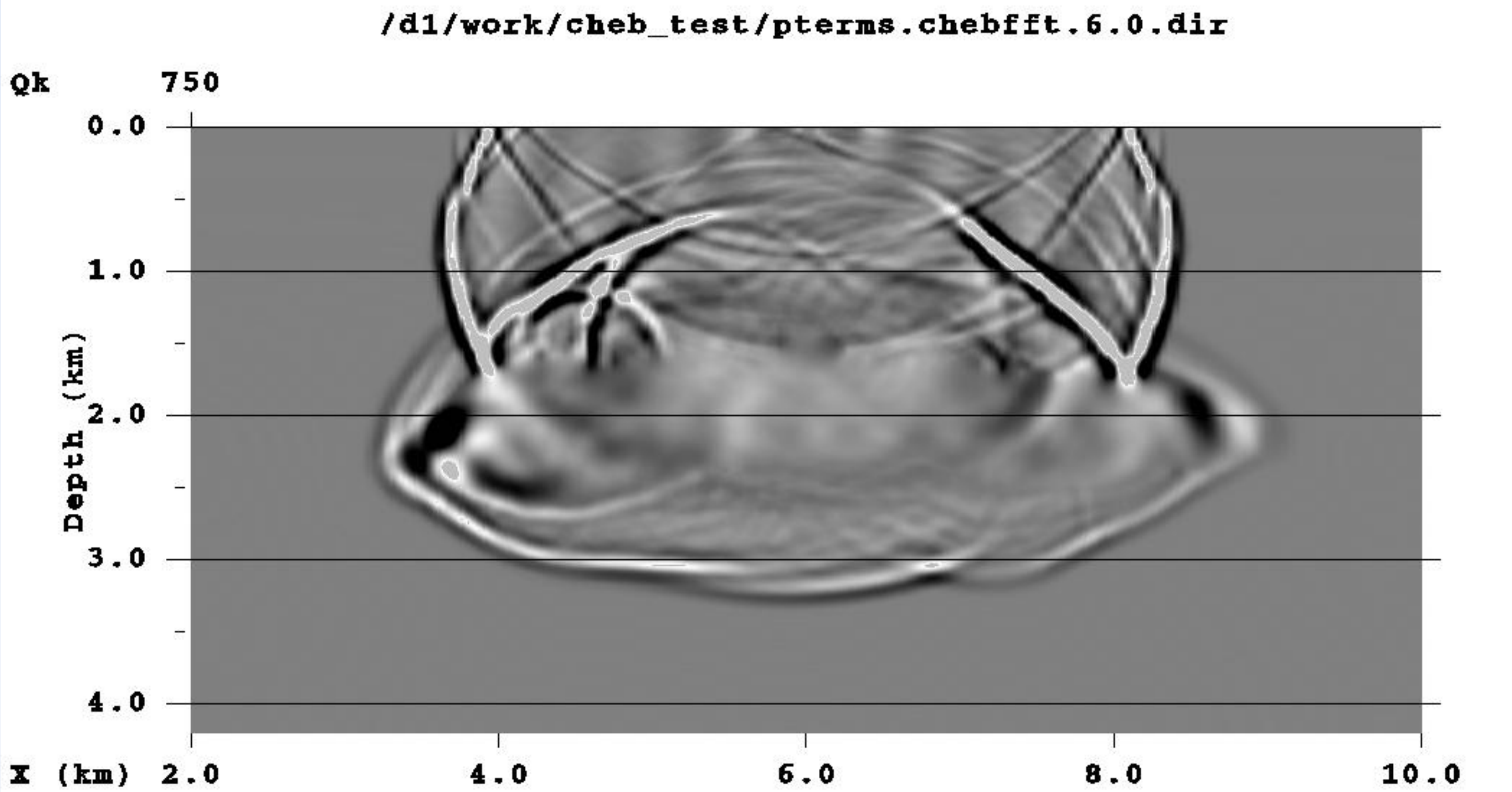


Imaging beneath salt is difficult because the waves we use to probe the salt body and the structures beneath it follow complicated paths: Diffractions, multiple arrivals, refractions and prism waves are difficult to model using ray based and one way wave equation methods.



Two way solutions based on explicit time marching can model these complicated waves but their accuracy is dependent on the wave equation selected and its numerical implementation. Numerical implementations methods include: finite differences, optimized convolutional operators and pseudo spectral methods for the spatial operators.

Given the difficulties in imaging salt bodies and the structures beneath due to

complex wave paths

uncertainties in the velocity model

limited: apertures, bandwidth & coverage during the data acquisition

it should be clear that

we want to use the most accurate numerical method possible so we can eliminate uncertainties in the imaging process due solely to the numerical methods employed

we also know:

computers continue to get faster and less expensive making more accurate numerical methods economically feasible

In this paper we address the numerical issues for seismic modeling and RTM using a 2 way wave equation propagator:

what should we use for the spatial operators ?

finite differences
optimized convolutional operators
pseudo spectral methods

what is the best method to advance the wave fields in time ?

2nd order finite differences
higher order finite differences e.g. Lax-Wendroff
rapid expansion method

so that we can minimize numerical errors and reduce uncertainties and numerical noise in the RTM imaging process.

Acoustic wave equation - An exact solution

The 2-D acoustic acoustic wave equation

$$\frac{\partial^2 p}{\partial t^2} = -L^2 p + S \tag{1}$$

The operator $-L^2$ is given by:

$$-L^2 = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \tag{2}$$

The initial conditions:

$$\frac{\partial p}{\partial t}(t = 0) = \dot{p}_0 \qquad and \qquad p(t = 0) = p_0 \tag{3}$$

The basis for the REM method is the formal solution of equation (1), with the initials conditions.

Laplacian Evaluation

$$-L^2 = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)$$

pseudo spectral operator: $$\frac{\partial^2 f}{\partial x^2} = IFFT\left[-k_x^2 FFT\left[f(x)\right]\right]$$

truncated Taylor series expansion leading to
standard finite difference, *D*, operators 4th, 6th , 8th order etc.

optimized finite difference operators based on
finite impulse response, FIR, operator 4th, 6th, 8th order etc

the last two can be viewed as a convolution of the wave field
with the operator in space

Convolutional Finite Difference Operators
--- 2nd-order derivative

$$\frac{\partial^2}{\partial x^2} \Leftrightarrow -k_x{}^2$$ ⟸ convolution theorem

$$d_2(x) = \frac{1}{2\pi}\int_{-k_N}^{k_N} -k_x{}^2 e^{ik_x x} dk_x$$ ⟸ inverse Fourier transform

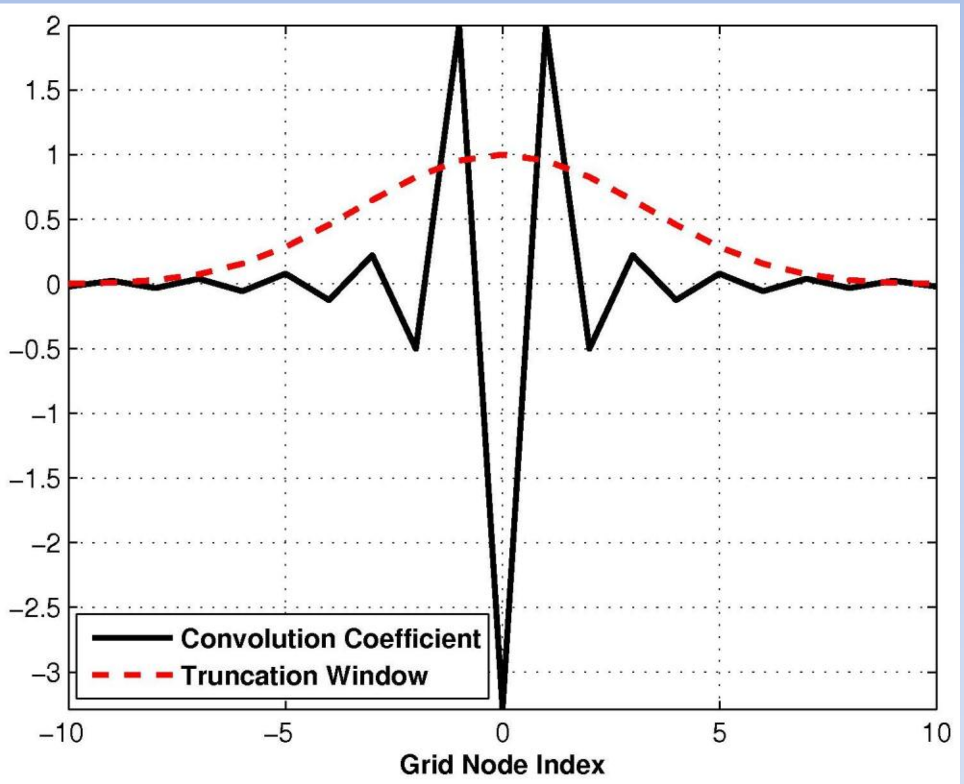
$k_N = \frac{\pi}{\Delta x}$ is the Nyquist frequency.

$$d_2(x) = \begin{cases} -\frac{k_N{}^3}{3\pi} & \text{for } x = 0 \\ -\frac{1}{\pi x}\left\{\left[k_N{}^2 - \frac{2}{x^2}\right]\sin\left(k_N x\right) + \frac{2k_N}{x}\cos\left(k_N x\right)\right\} & \text{for } x \neq 0 \end{cases}$$

$$\boxed{d_2(x) = \begin{cases} -\frac{\pi^2}{3\Delta x^3} & \text{for } x = 0 \\ -\frac{2}{n^2\Delta x^3}\left(-1\right)^n & \text{for } x \neq 0 \end{cases}}$$ ⟸ discrete convolution coefficients

Zhou & Greenhalgh, 1992

FIR Convolution Coefficients



2nd-order derivative on standard grids

2nd -order derivative on regular grids is replaced with a convolutional Finite Impulse Response filter

FIR(n) = *d*2(n)**w*(n)

w(n) is a Hanning tapered version of the standard operator *d*2(n). N is the half truncation width, alpha = .54 & beta = 6.0

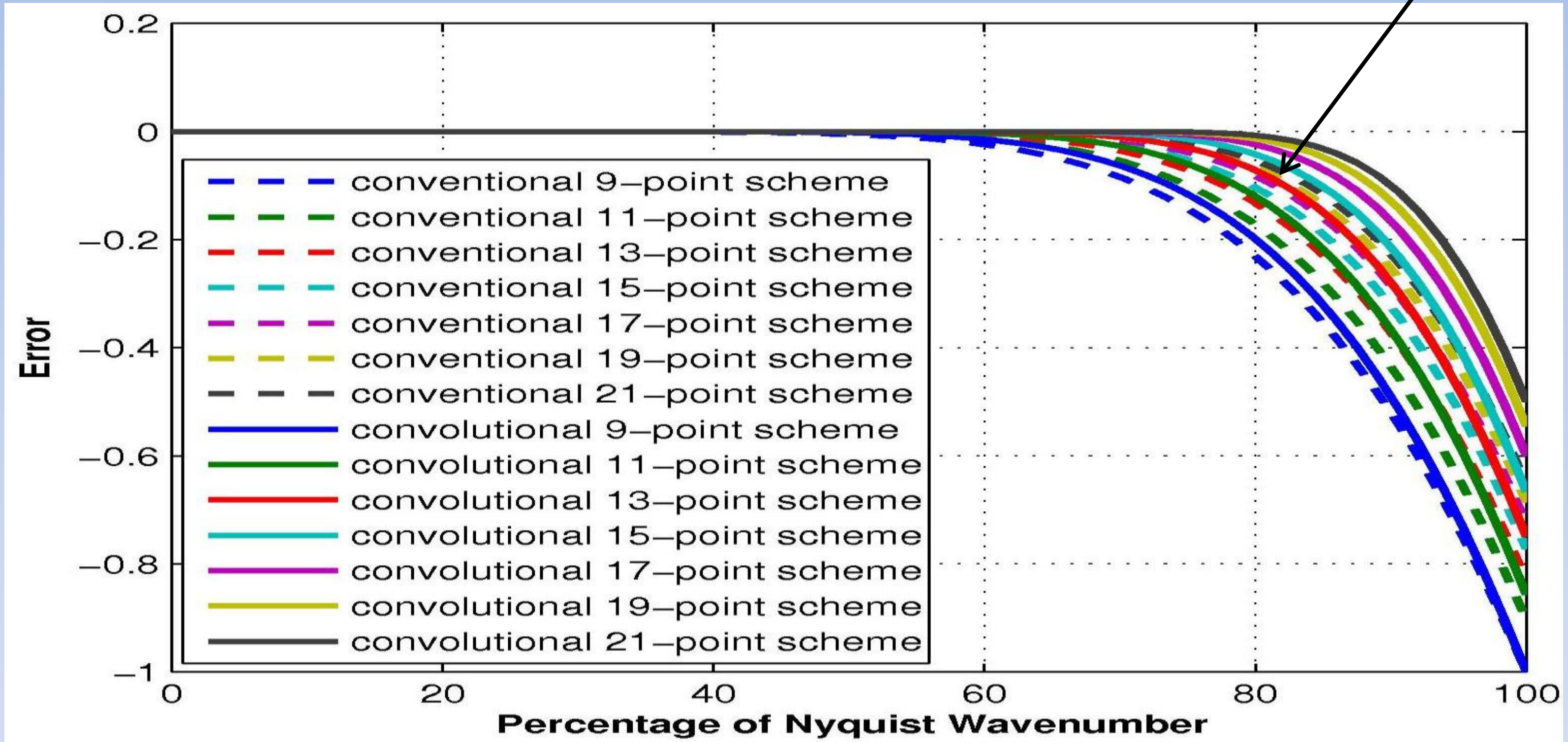
$$w(n) = \left[2\alpha - 1 + 2(1-\alpha)\cos^2 \frac{n\pi}{2(N+2)} \right]^{\frac{\beta}{2}}$$

Zhou and Greenhalgh (1992)

After Chu et al., 2009

Dispersion Curves

13 point FIR used



2nd-order derivative on standard grids: FIR vs. truncated Taylor

Chu et al., 2009

now we compare the 3 Laplace operators:

pseudo spectral – fft

4th order finite differences – convolution

13 point FIR – convolution

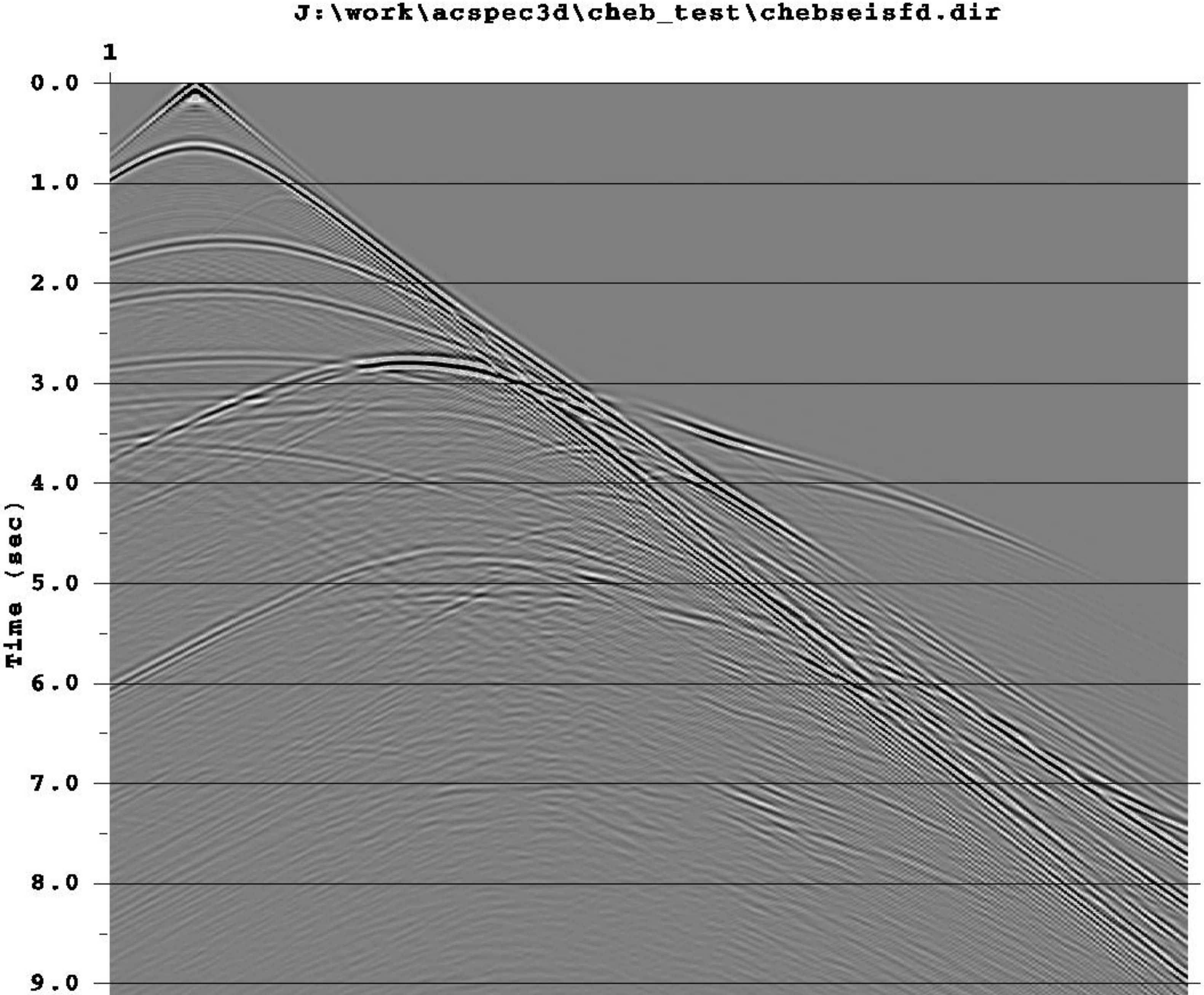
to record long offsets we place the source at 1.0 km

we record 10 seconds of data

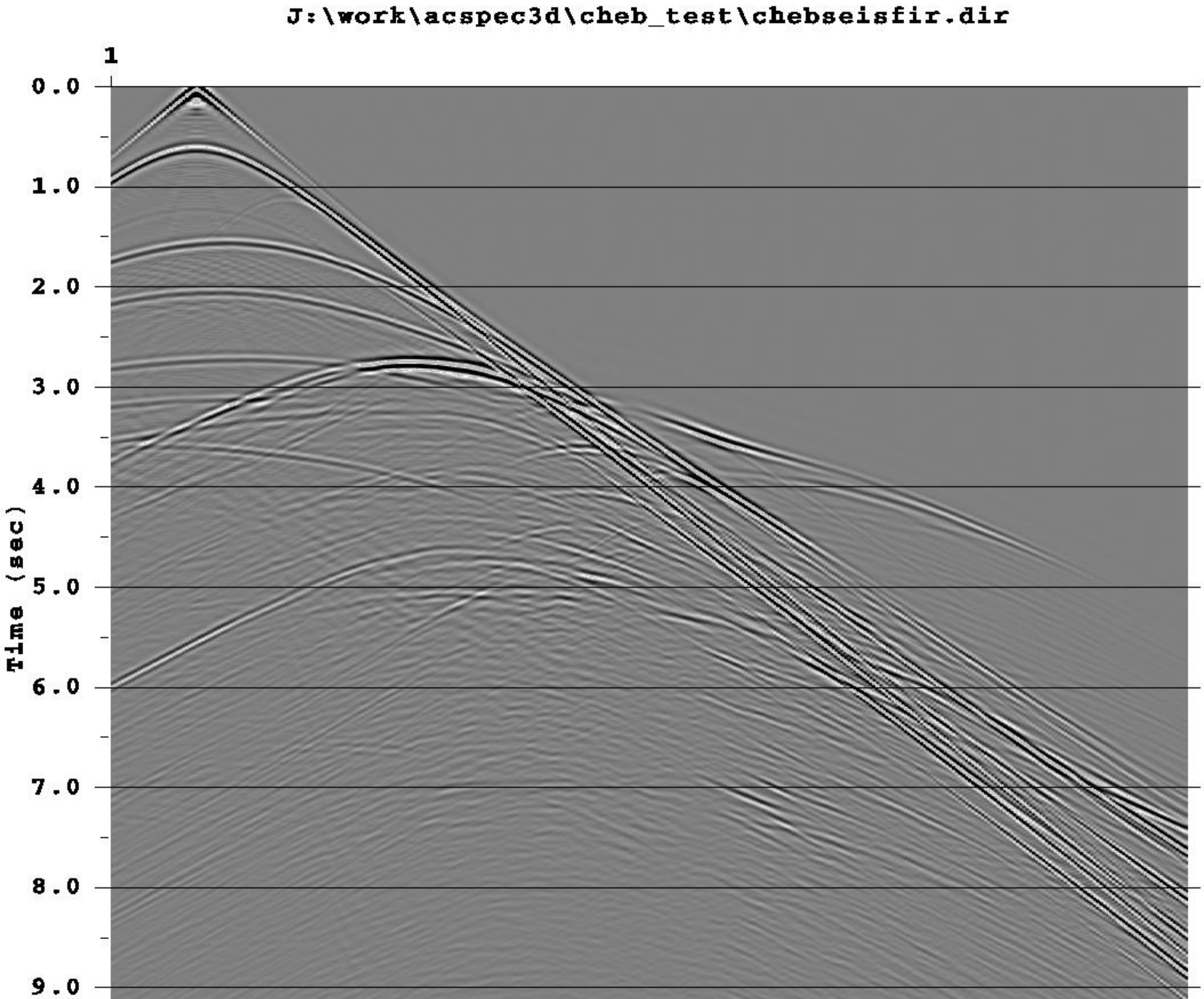
the sample rate was .008s

all other model parameters remain the same

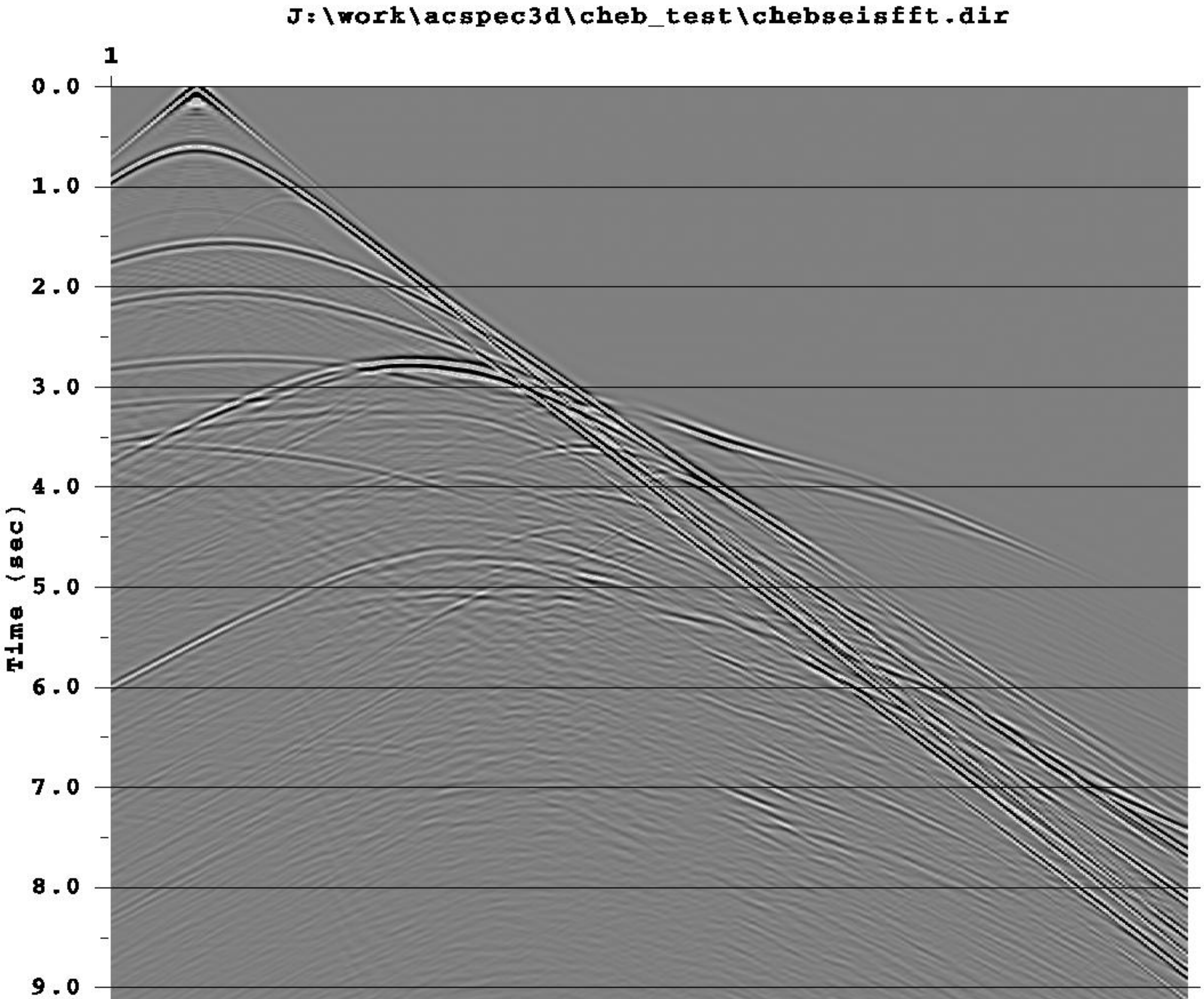
One Step REM and 4th order FD



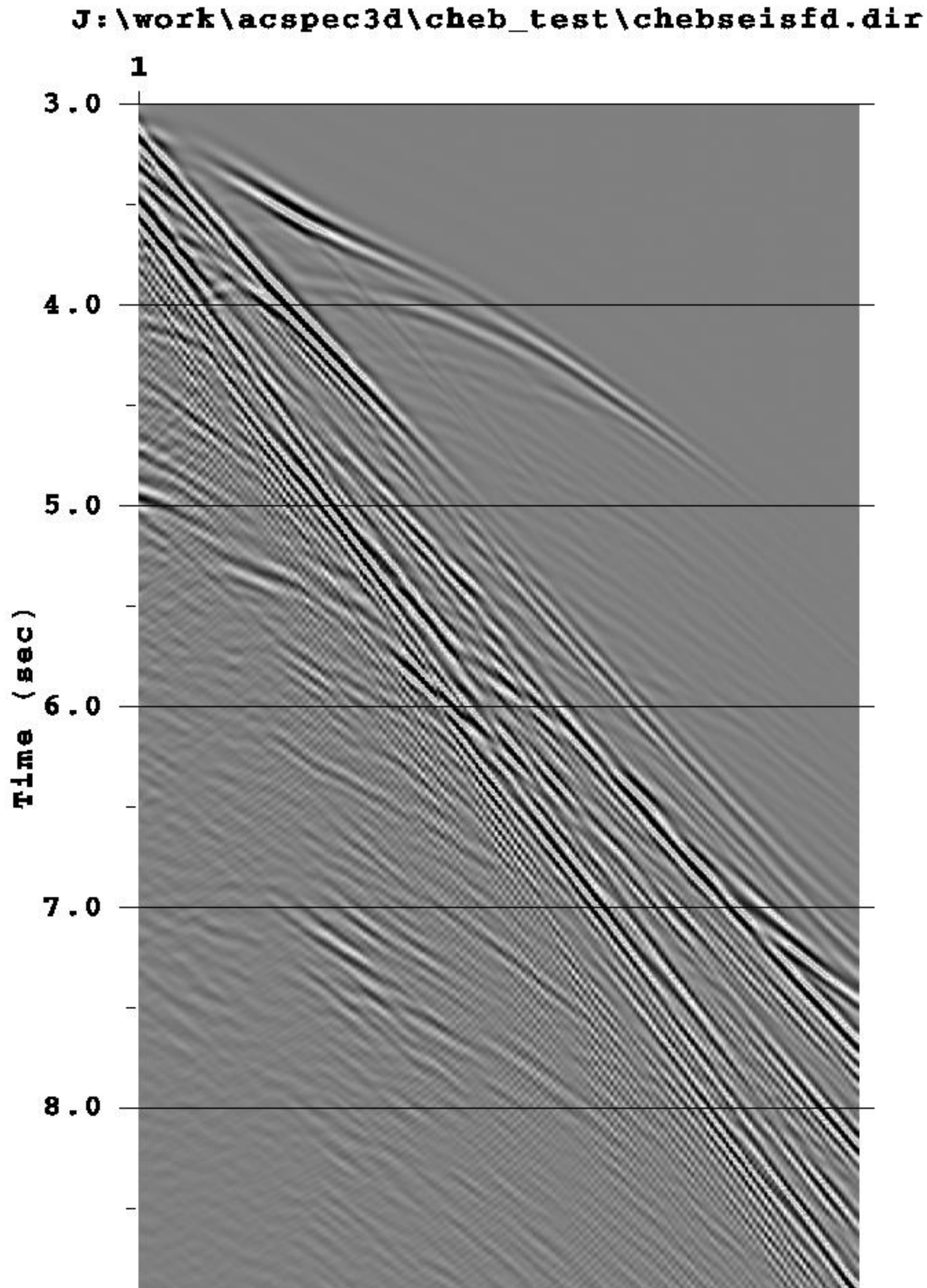
One Step REM and 13 pt FIR

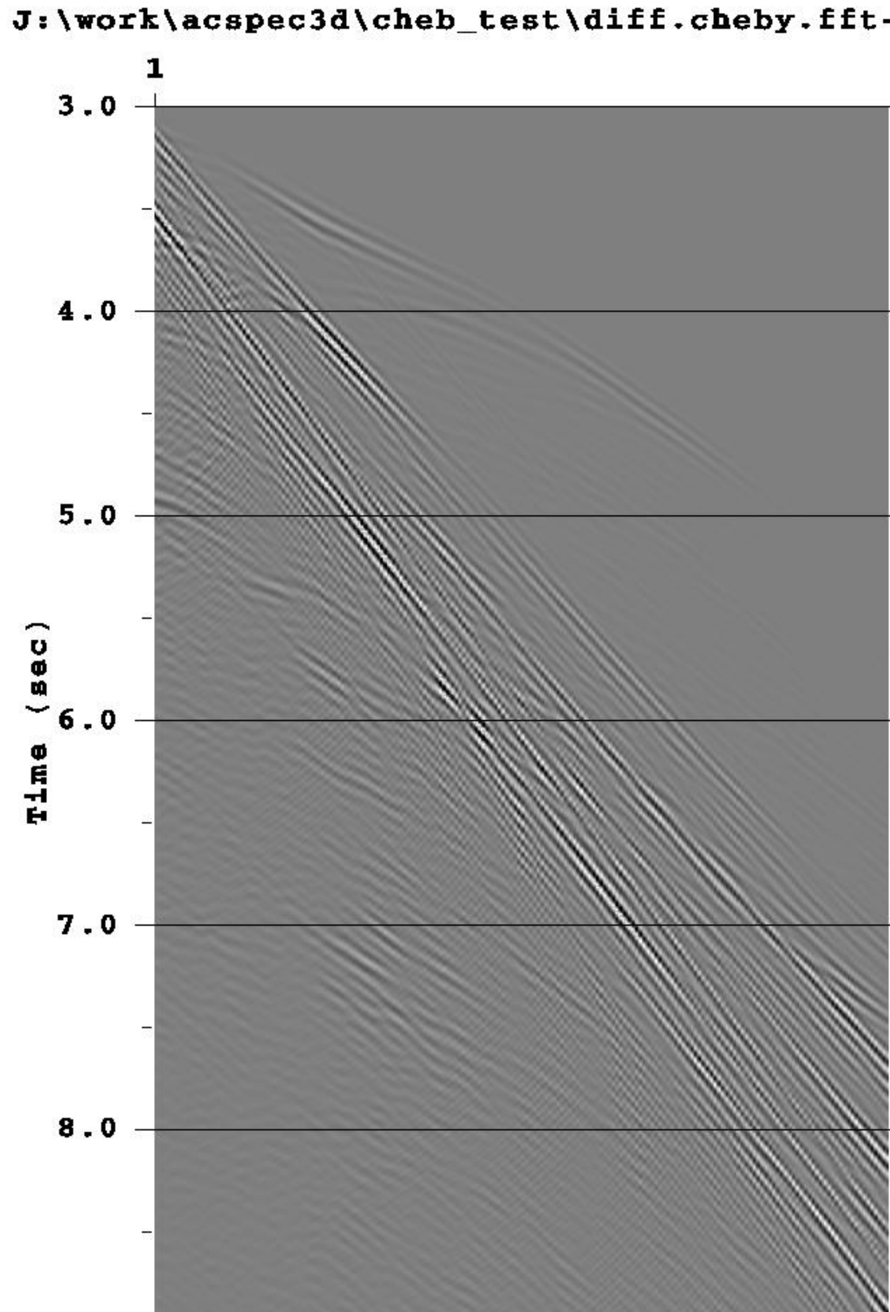


One Step REM and pseudo spectral

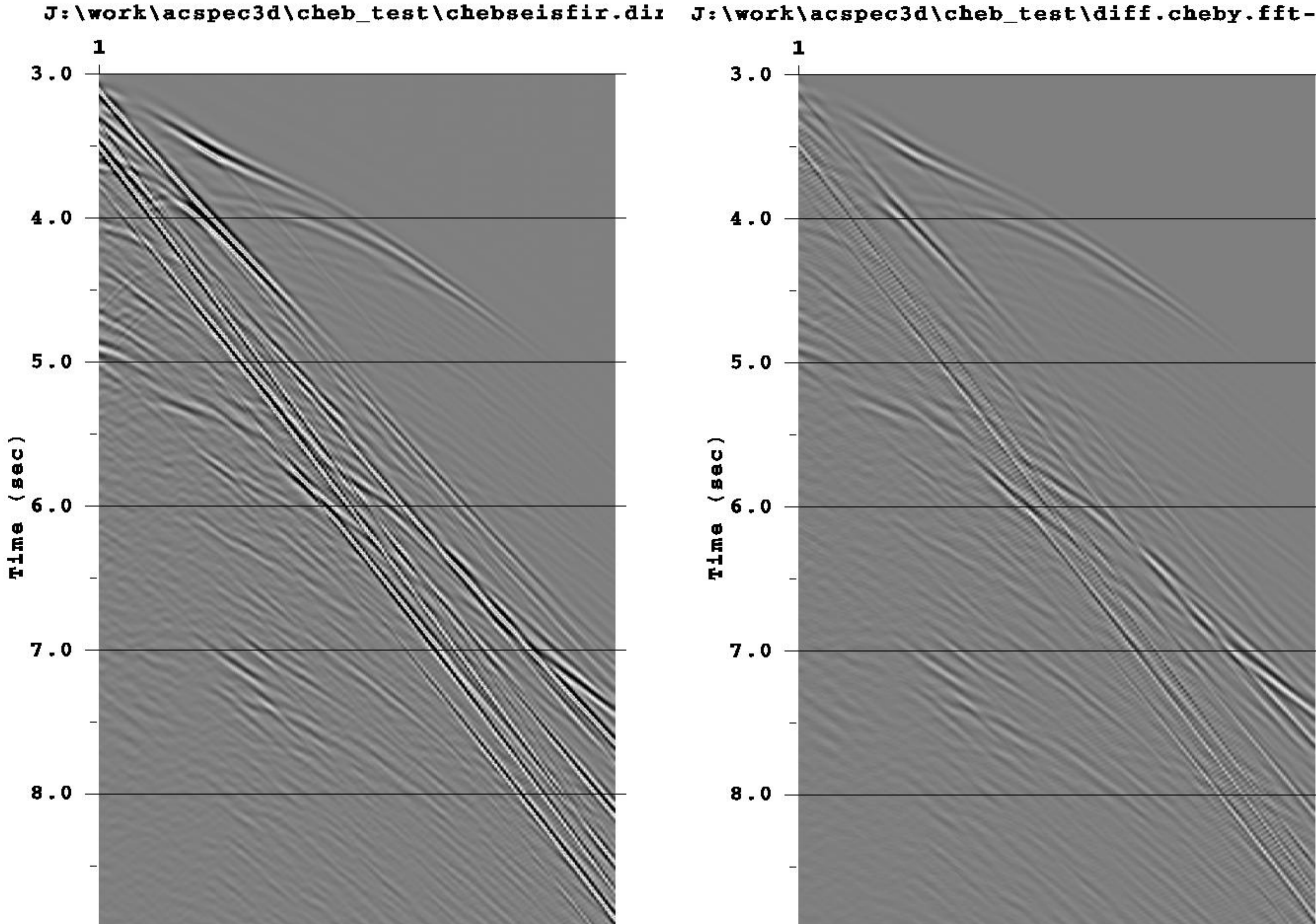


One Step REM and *d*4 and FFT – *d*4

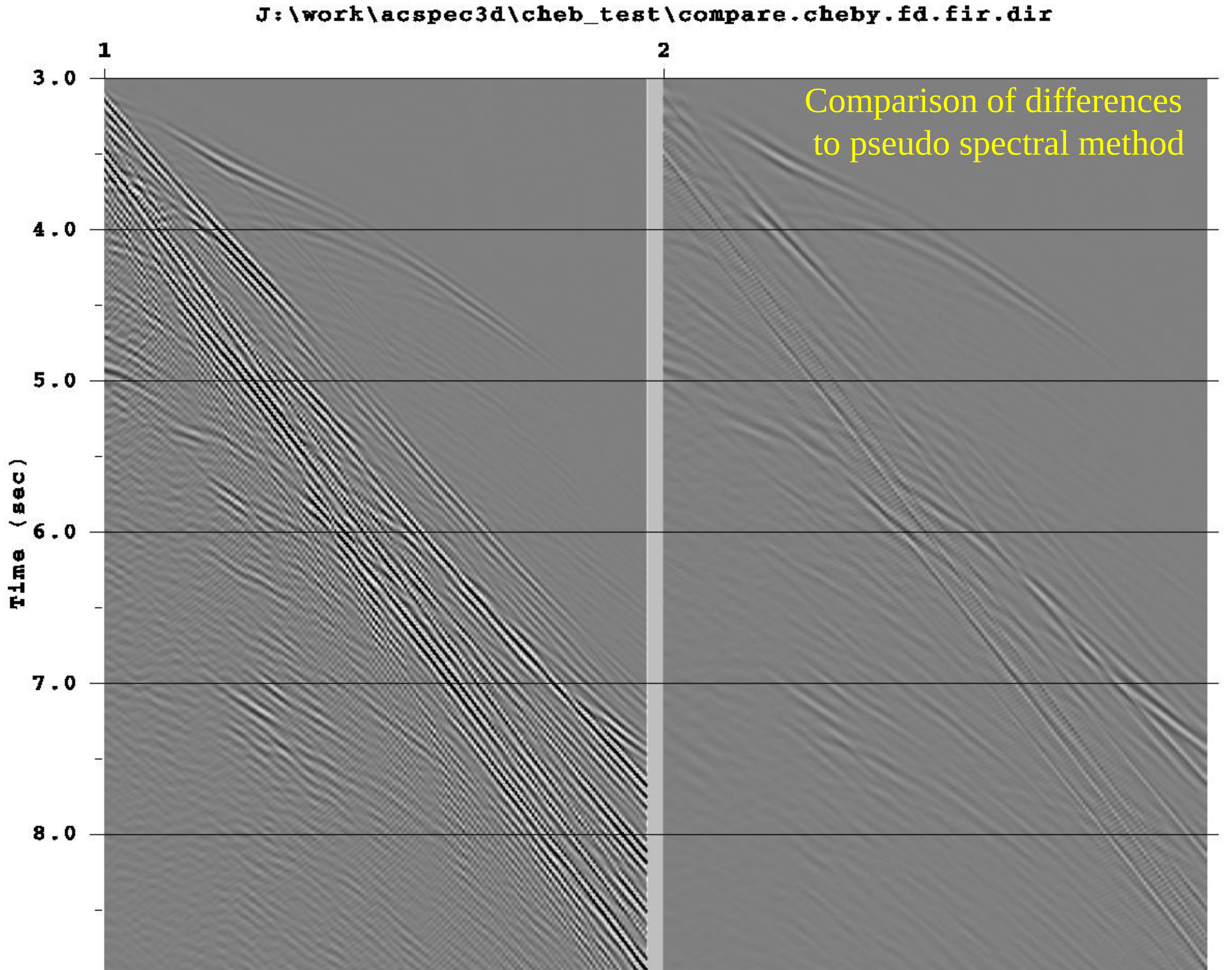




One Step REM and FIR and FFT - FIR



One Step REM and FFT – *d*4 and FFT - FIR



Now for the time operator:

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The basis for the REM method is the formal solution of equation (1), with the initials conditions.

Rapid expansion method

The formal solution of wave equation with the initial conditions:

$$p_t = \cos(L\,t)\,p_0 + \frac{sin(L\,t)}{L}\,\dot{p}_0 \qquad (4)$$

Adding to (4) the solution for the term p_{-t} then we get

$$p_t = -p_{-t} + 2\,\cos(L\,t)\,p_0 \qquad (5)$$

However, the cosine function in the equation (5) can be also written as:

$$\cos(L\,t) = \frac{1}{2}\left(e^{-i\,L\,t} + e^{i\,L\,t}\right). \qquad (6)$$

Rapid expansion method - cont

The REM is obtained with the following expansion of the cosine function [Tal-Ezer, 1986]:

$$\cos(L\,\Delta t) = \sum_{k=0;k\ even}^{M} C_k\, J_k(t\,R)\, Q_k\left(\frac{i\,L}{R}\right) \tag{7}$$

where $C_0 = 1$ and $C_k = 2$ for $k \geq 1$. J_k represents the Bessel function of order k and $Q_k(w)$ are modified Chebyshev polynomials.

The recursion for $Q_k(w)$ with only even terms

$$Q_{k+2}(w) = (4w^2 + 2)\, Q_k(w) \;-\; Q_{k-2}\,(w). \tag{8}$$

The recursion is initiated by:

$$Q_0(w) = 1 \qquad and \qquad Q_2(w) = 1 + 2w^2.$$

Implementation of REM

For 2D wave propagation R is approximated given by

$$R = \pi c_{max} \sqrt{\frac{1}{dx^2} + \frac{1}{dz^2}}$$

with c_{max} the highest velocity in the grid and dx and dz are the grid spacing ([Kosloff et al., 1989]).

The sum in Tal-Ezer expansion is known to converge exponentially for $k > tR$ and, therefore, the summation can be safely truncated with k value slightly greater then $t\,R$.

One Step REM

$$p_t = \cos(Lt)\, p_0$$

Initial Condition: ***p0(x,y,z,t=0)=delta(x-xs, y-ys, z-zs, t=0)***

$$Q_0(w) = 1 \qquad and \qquad Q_2(w) = 1 + 2w^2.$$

Calculate Chebyshev Polynomials:

$$-L^2 = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)$$

apply Laplace Operator

for $k = 2,4, \ldots. M =$ tR

get next Chebyshev polynomial using recursion

$$Q_{k+2}(w) = (4w^2 + 2)\, Q_k(w) - Q_{k-2}(w).$$

Integrate Chebyshev Polynomials with *Jk(tR)*
save receiver data this time or
save snap shot data this time or
save both

for all times of interest

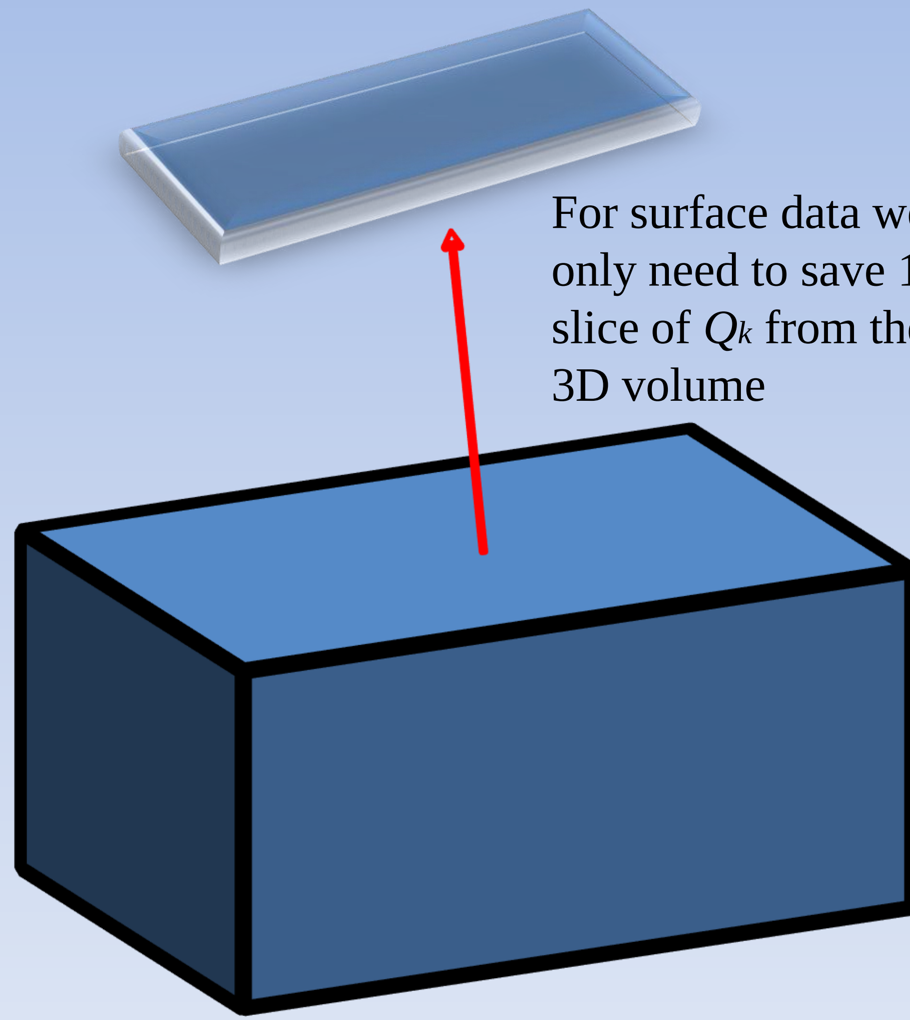
$$\sum_{k(even)}^{\infty} C_k\, J_k(tR)\, Q_k\left(\frac{iL}{R}\right)$$

$$Q_{k+2}(w) = (4w^2 + 2)\,Q_k(w) - Q_{k-2}(w).$$

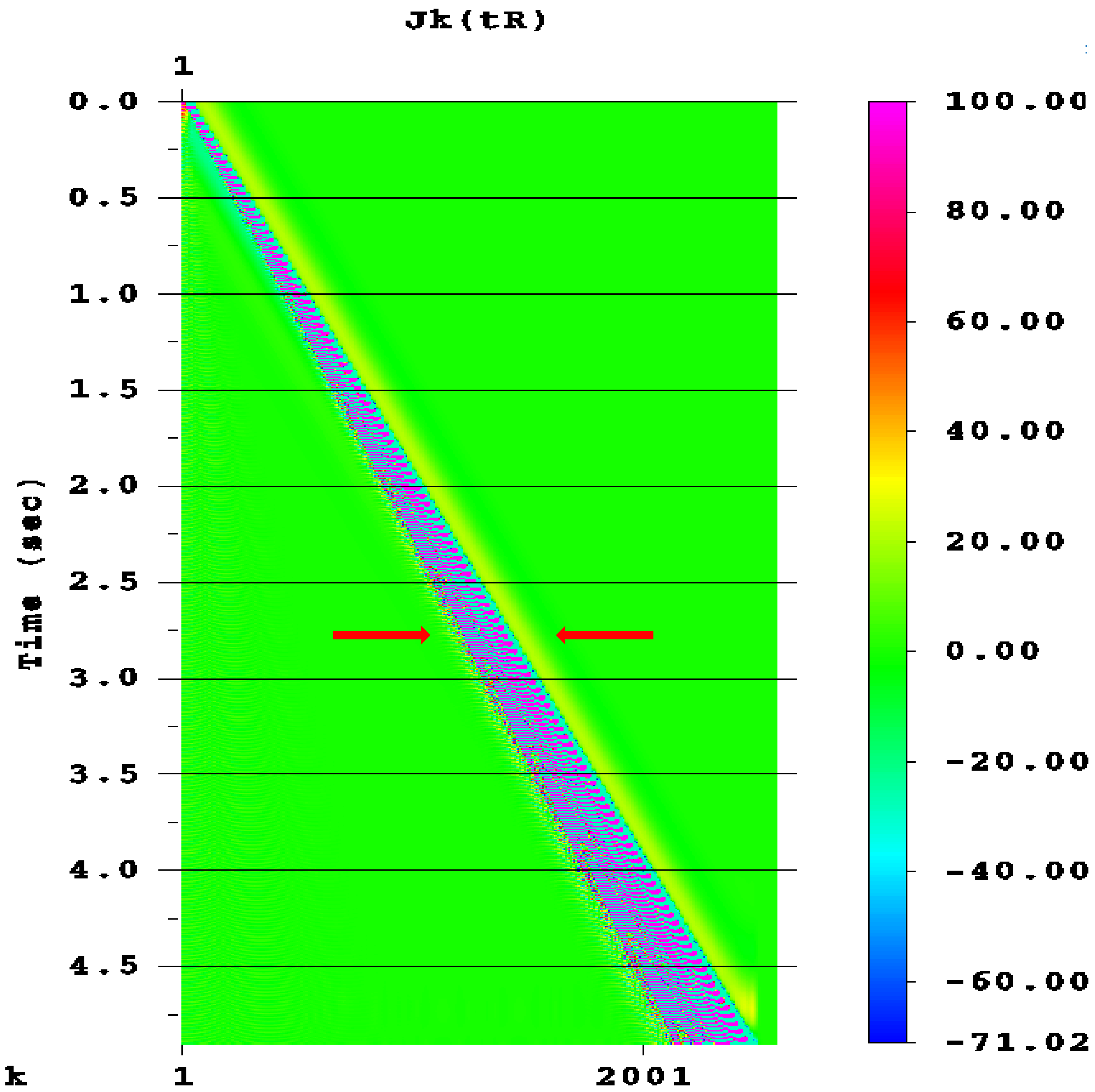
Q_k is defined and computed for the entire volume, but we only need to keep and integrate the values of Q_k where the receivers are located.

For example, if we are acquiring data at the surface, only the values of Q_k along the surface need to be saved for use in the integration.

This can greatly reduce the storage volume needed for 3D simulations.



$$\sum_{k(even)}^{\infty} C_k\, J_k(tR)\, Q_k\left(\frac{i\,L}{R}\right)$$



The integral can be limited based on the values of the Bessel Function at a given time.

R is defined from the spatial sample rate and the maximum velocity:

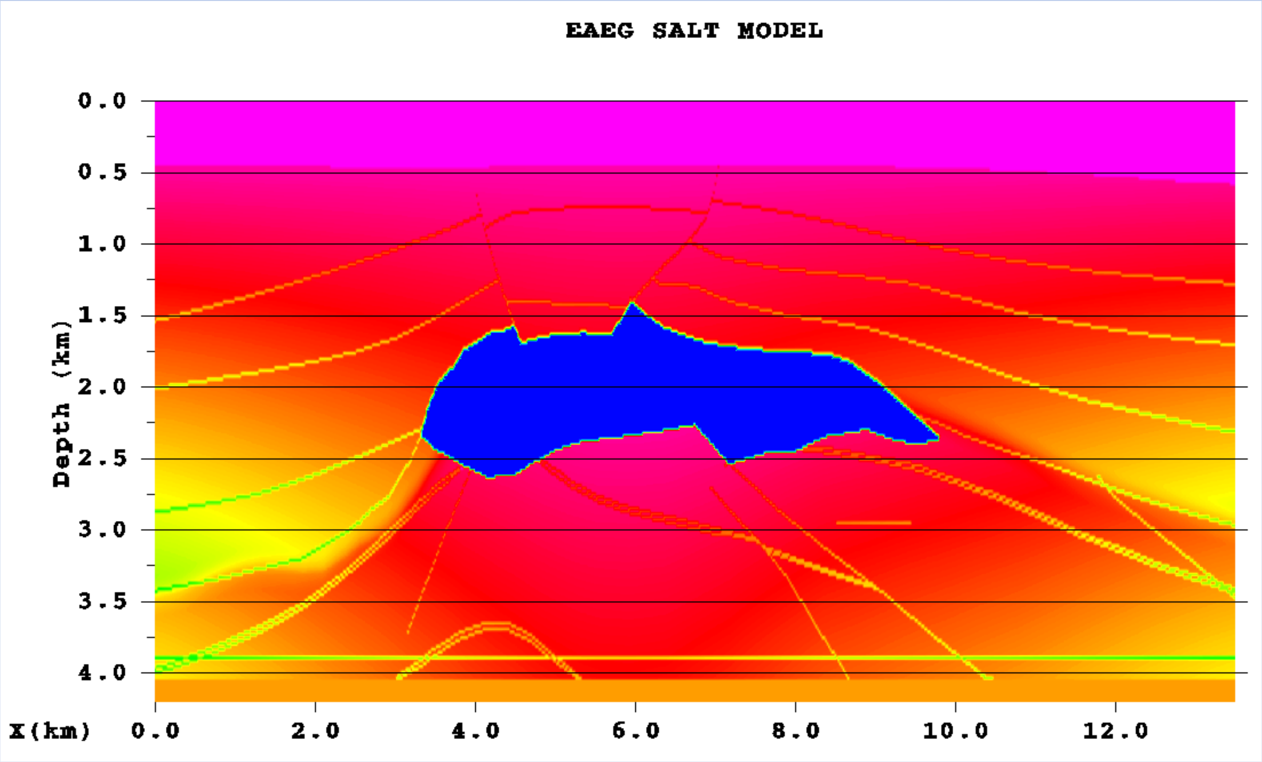
$$\pi c_{max}\left[\frac{1}{dx^2}+\frac{1}{dz^2}\right]^{1/2}$$

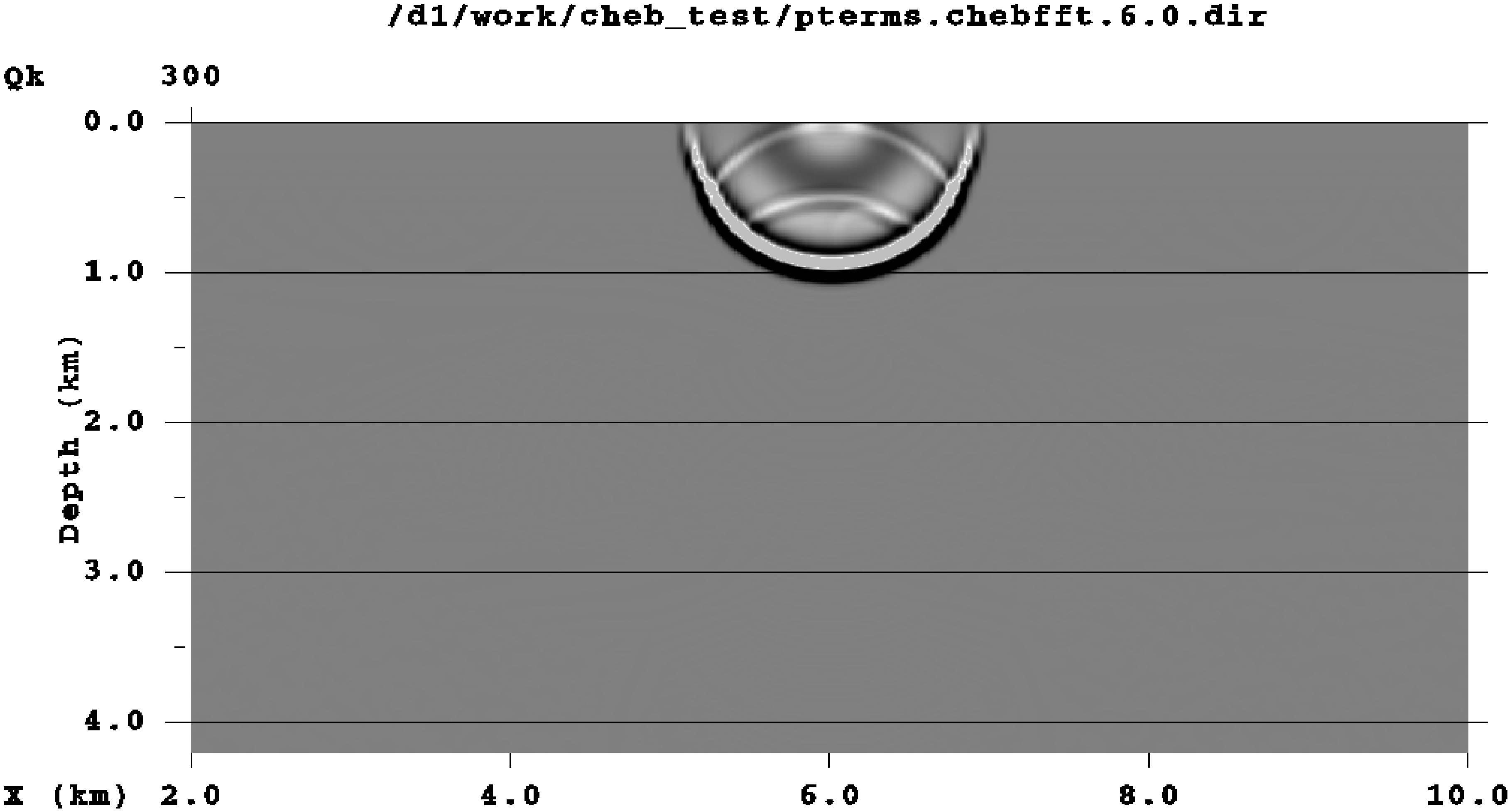
For values >tR the magnitude decreases quickly. Similarly a lower limit can be defined as a % of the line defining the maximum k needed.

now we show the Q_k Chebyshev polynomials

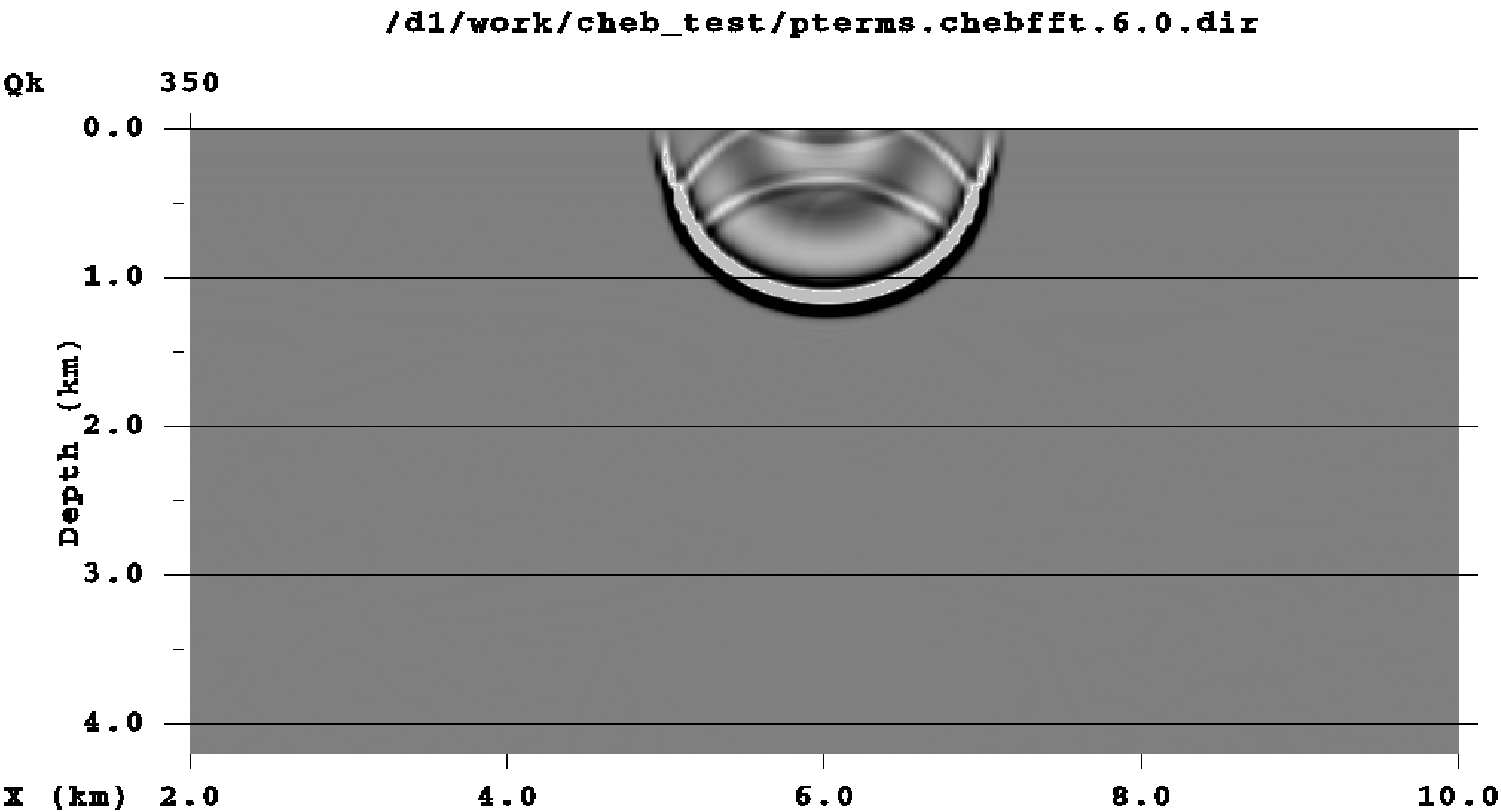
for k=300 to 750 every 50

we use the pseudo spectral or fft for the Laplacian
the source is a unit impulse response at 6.0 km
the sample rate was .008 s
we record 5.0 s of data
dx = dz = .02 km





/d1/work/cheb_test/pterms.chebfft.6.0.dir
Qk
300
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km)
2.0
4.0
6.0
8.0
10.0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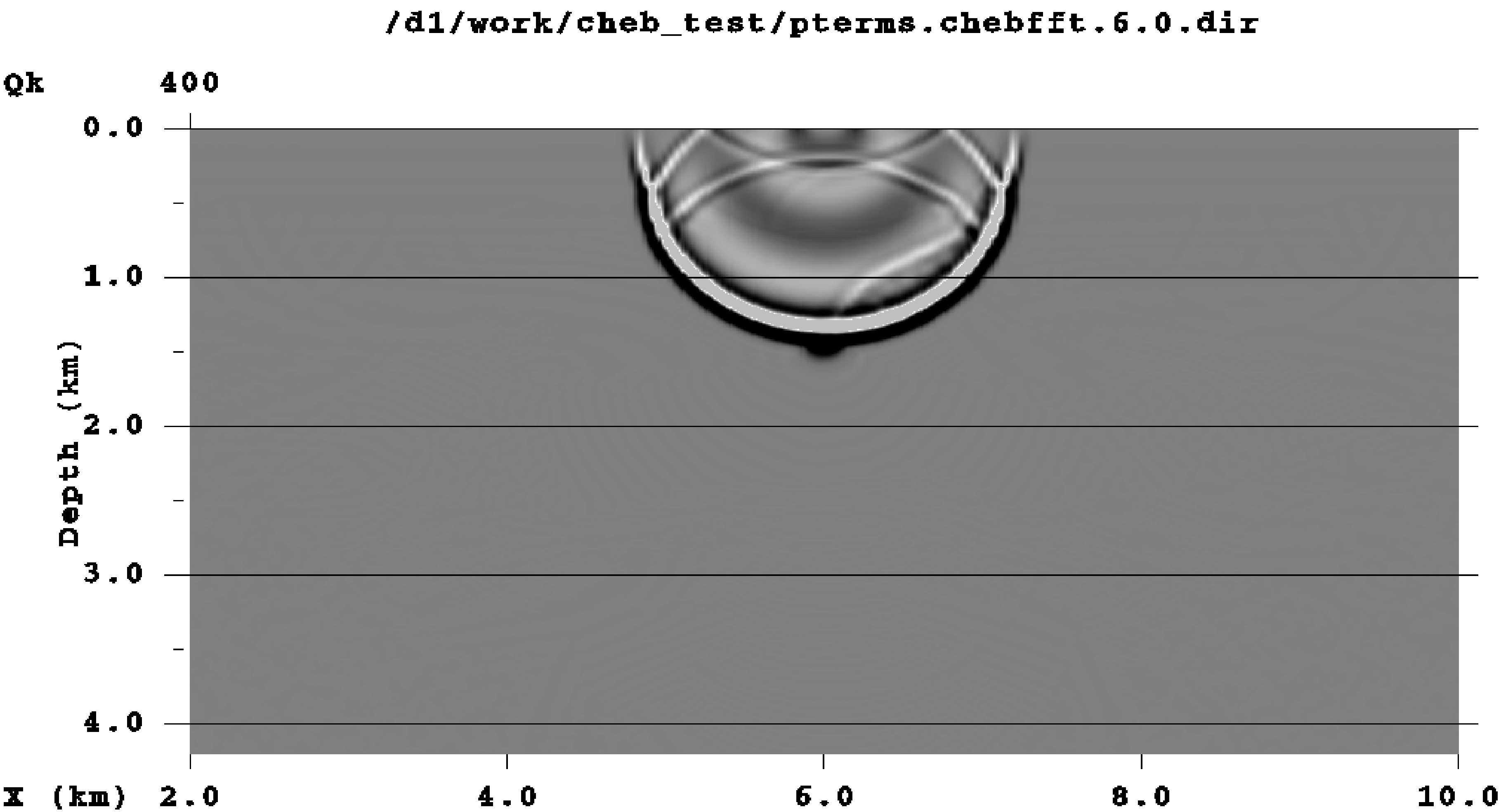
Qk 350
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0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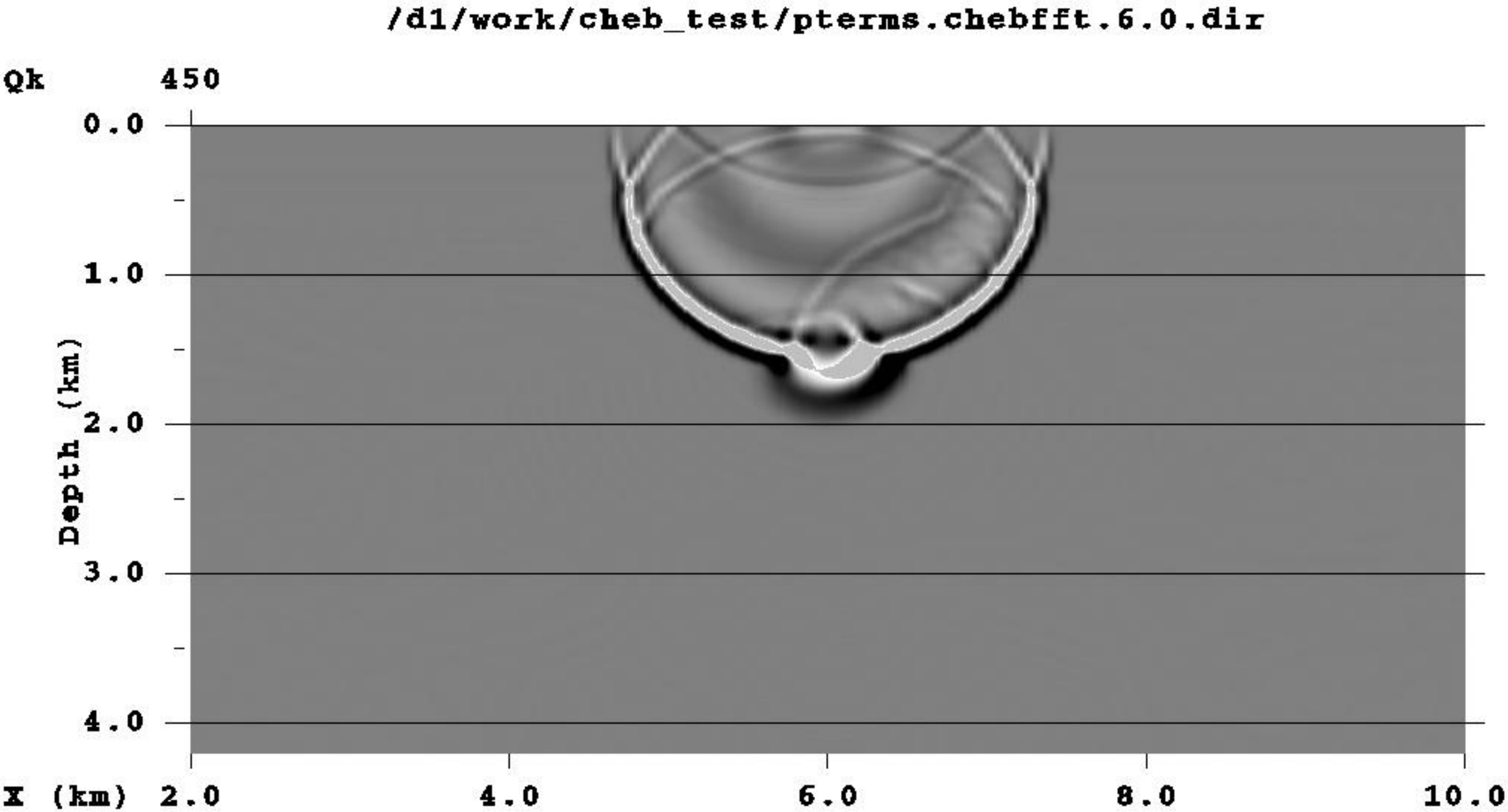
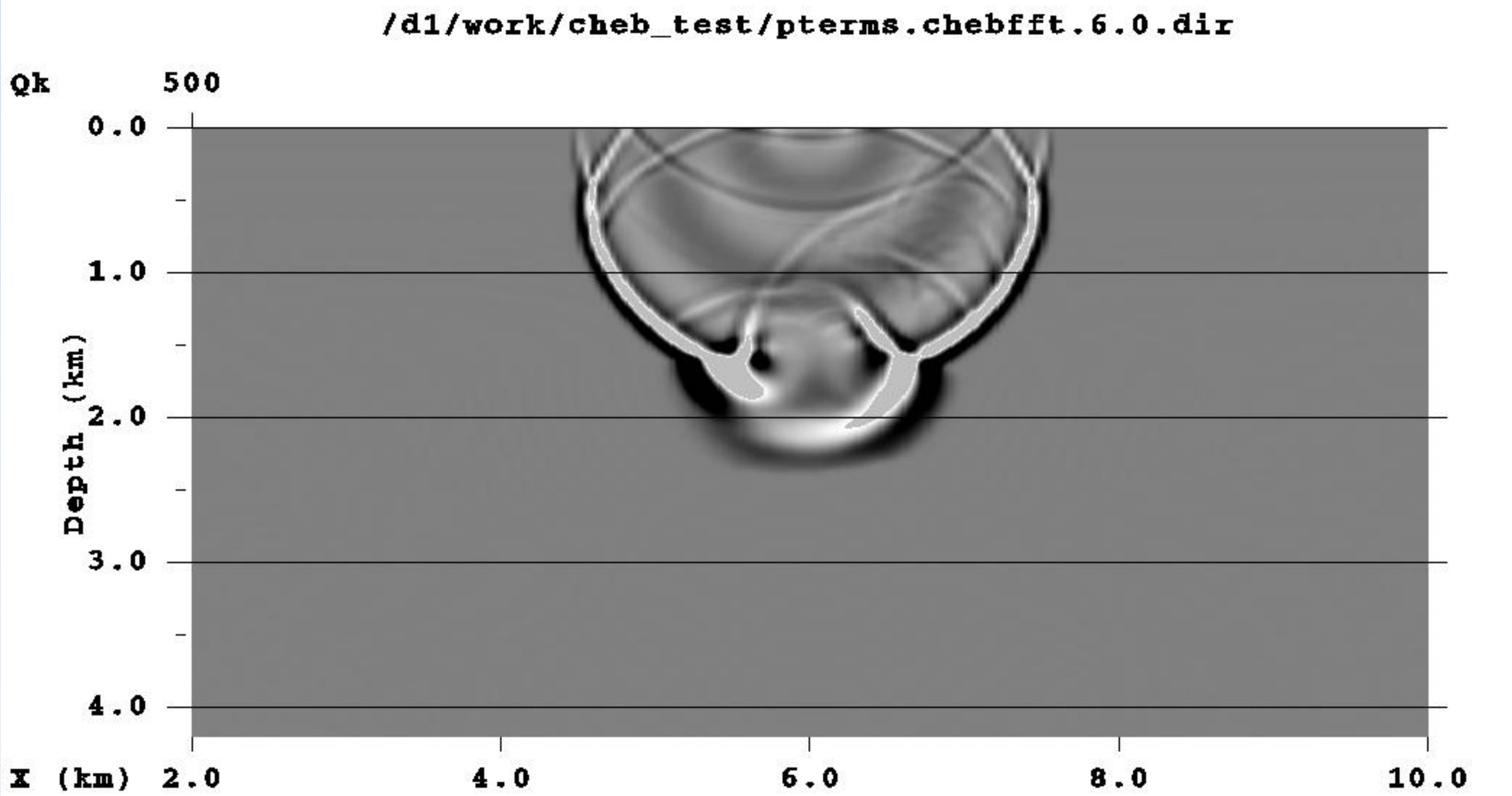
Qk 400
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0



/d1/work/cheb_test/pterms.chebfft.6.0.dir
Qk 450
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0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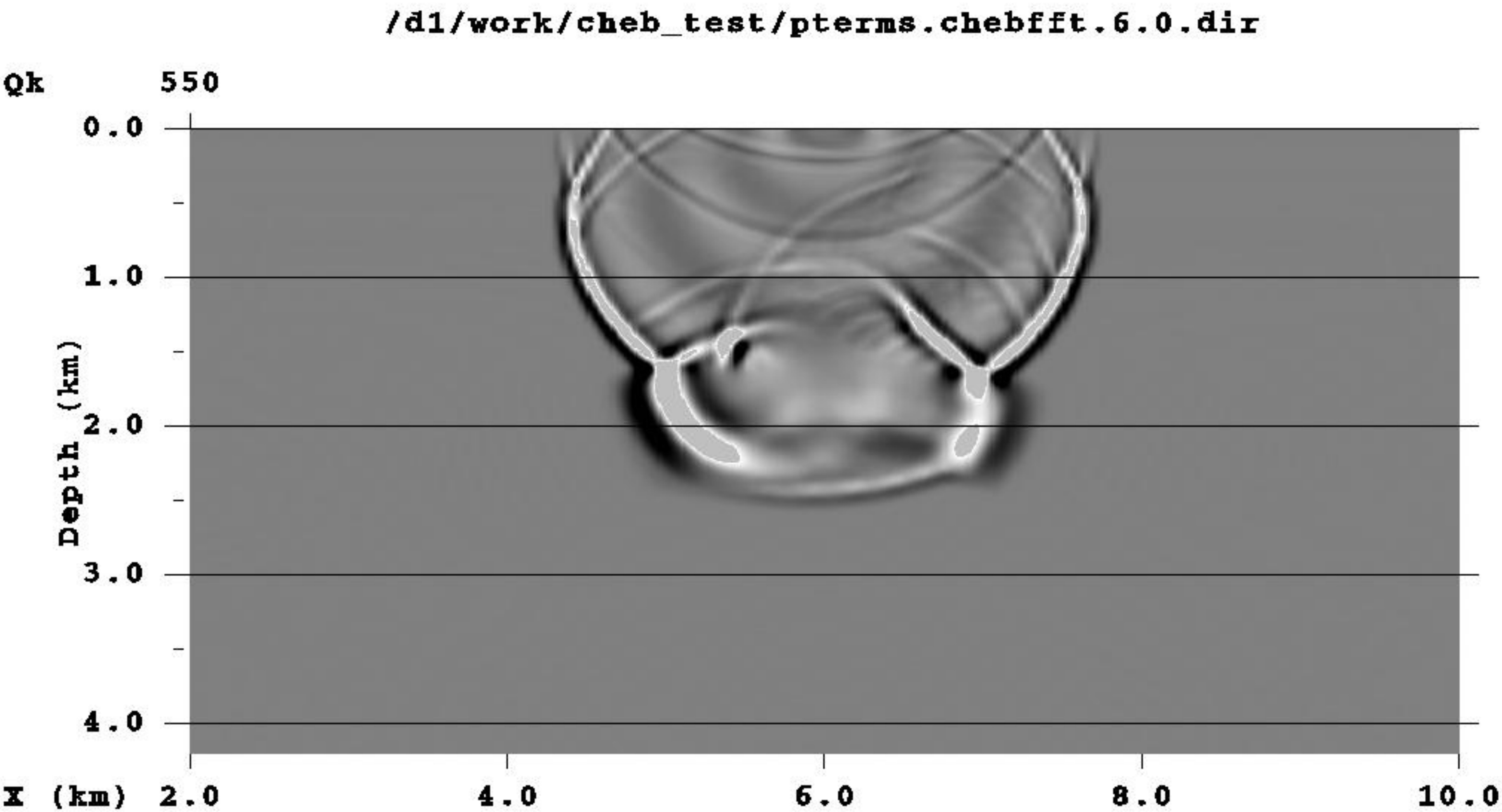
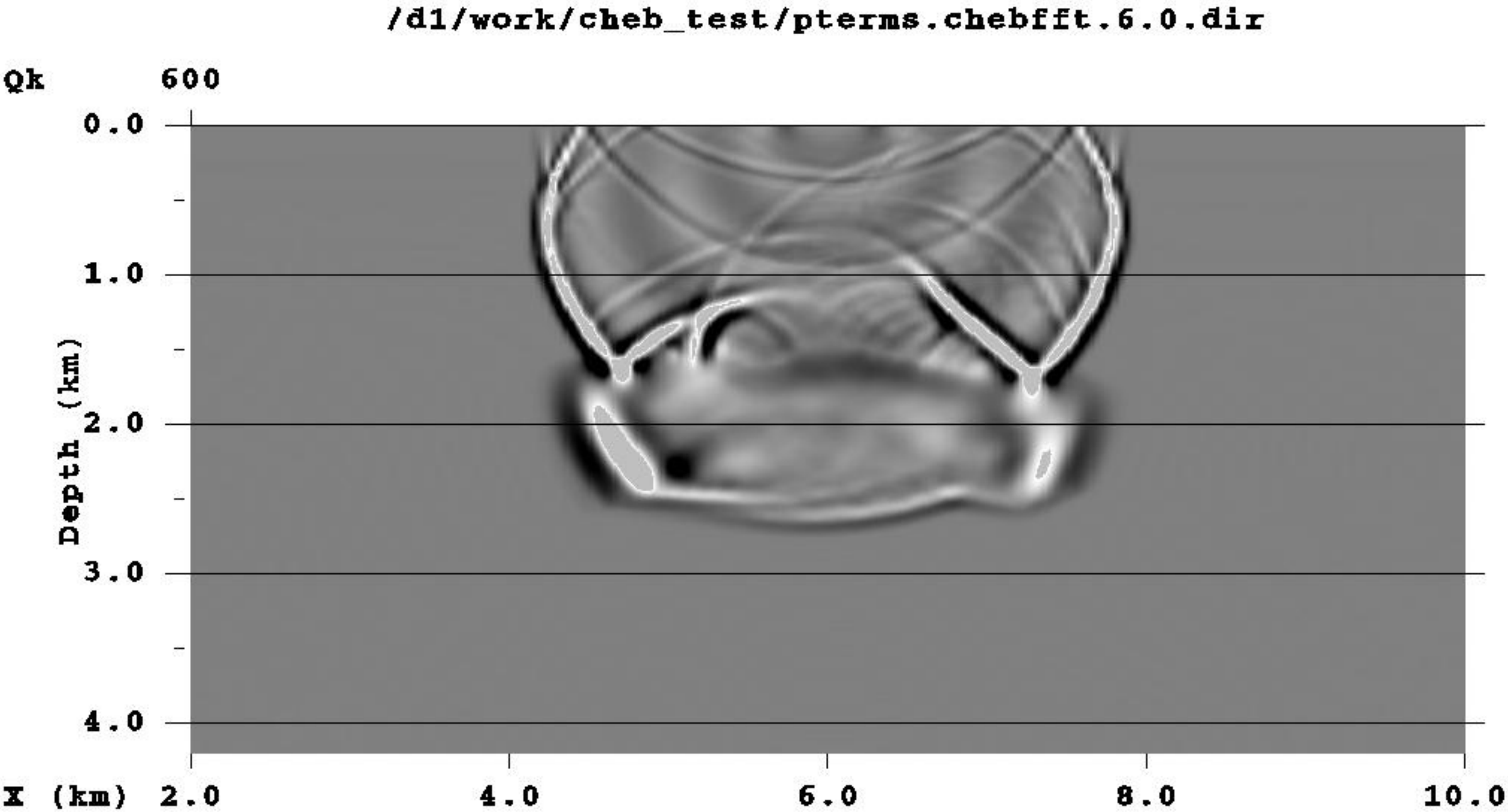
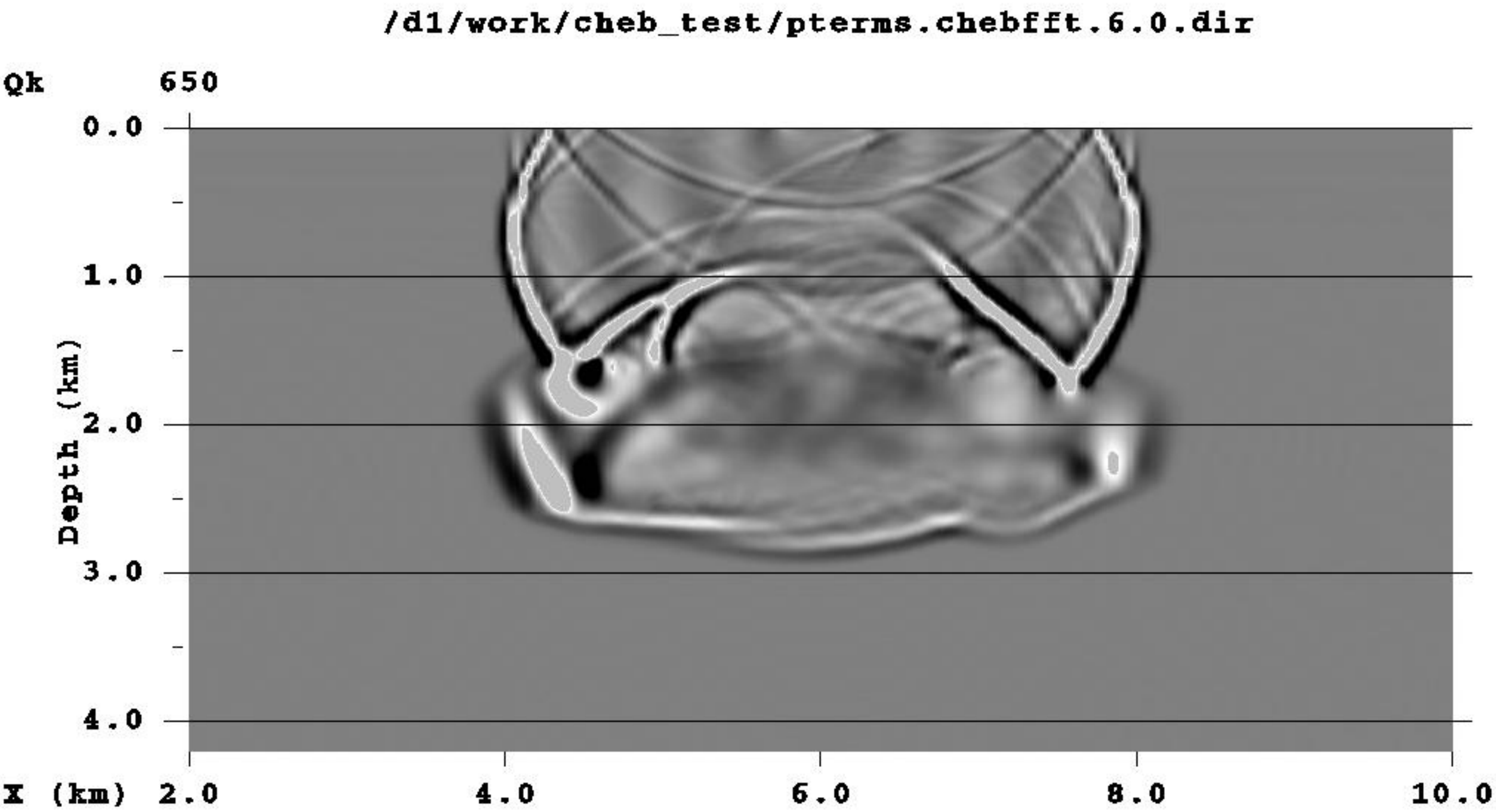
Qk 500
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0

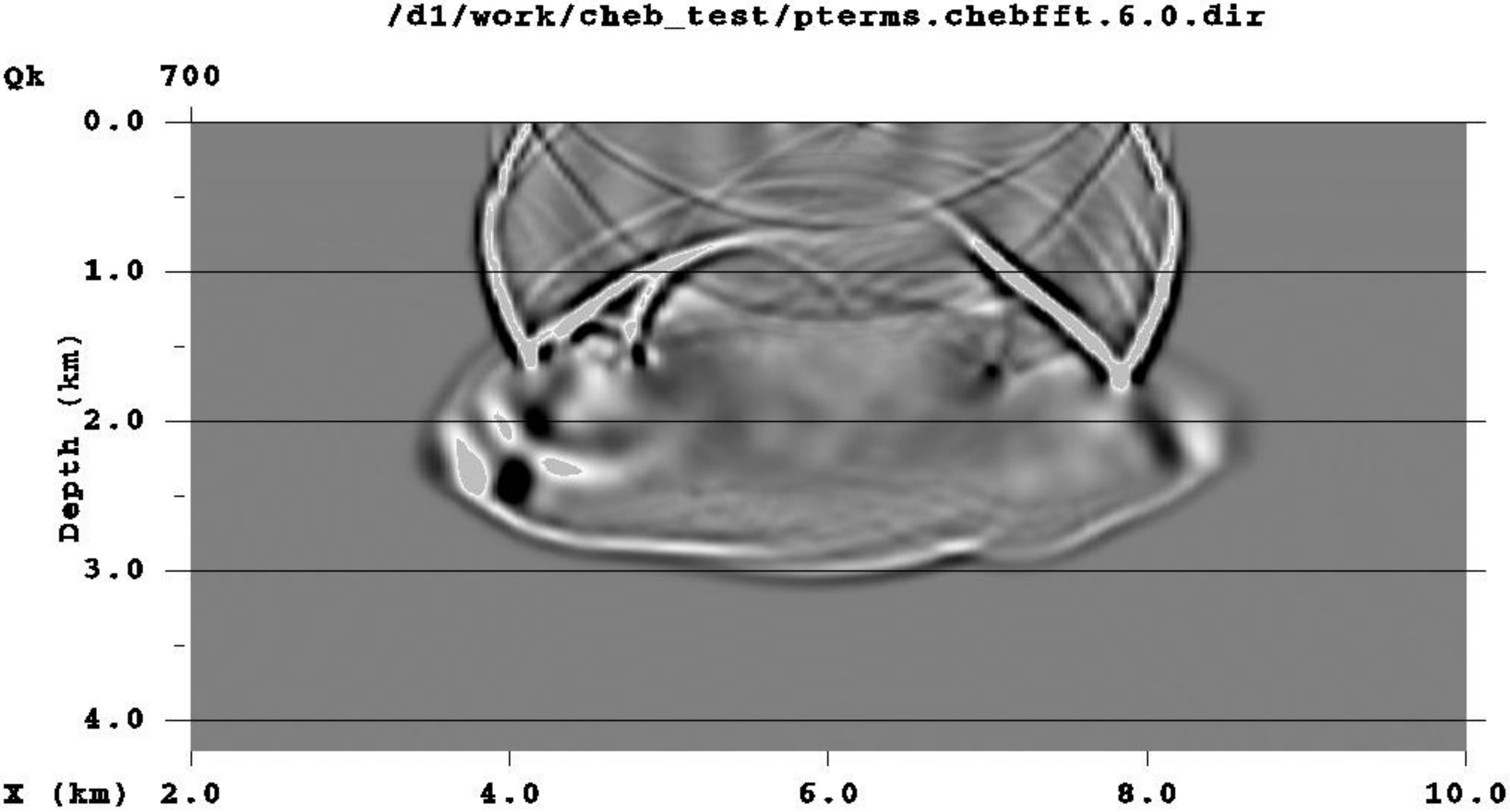


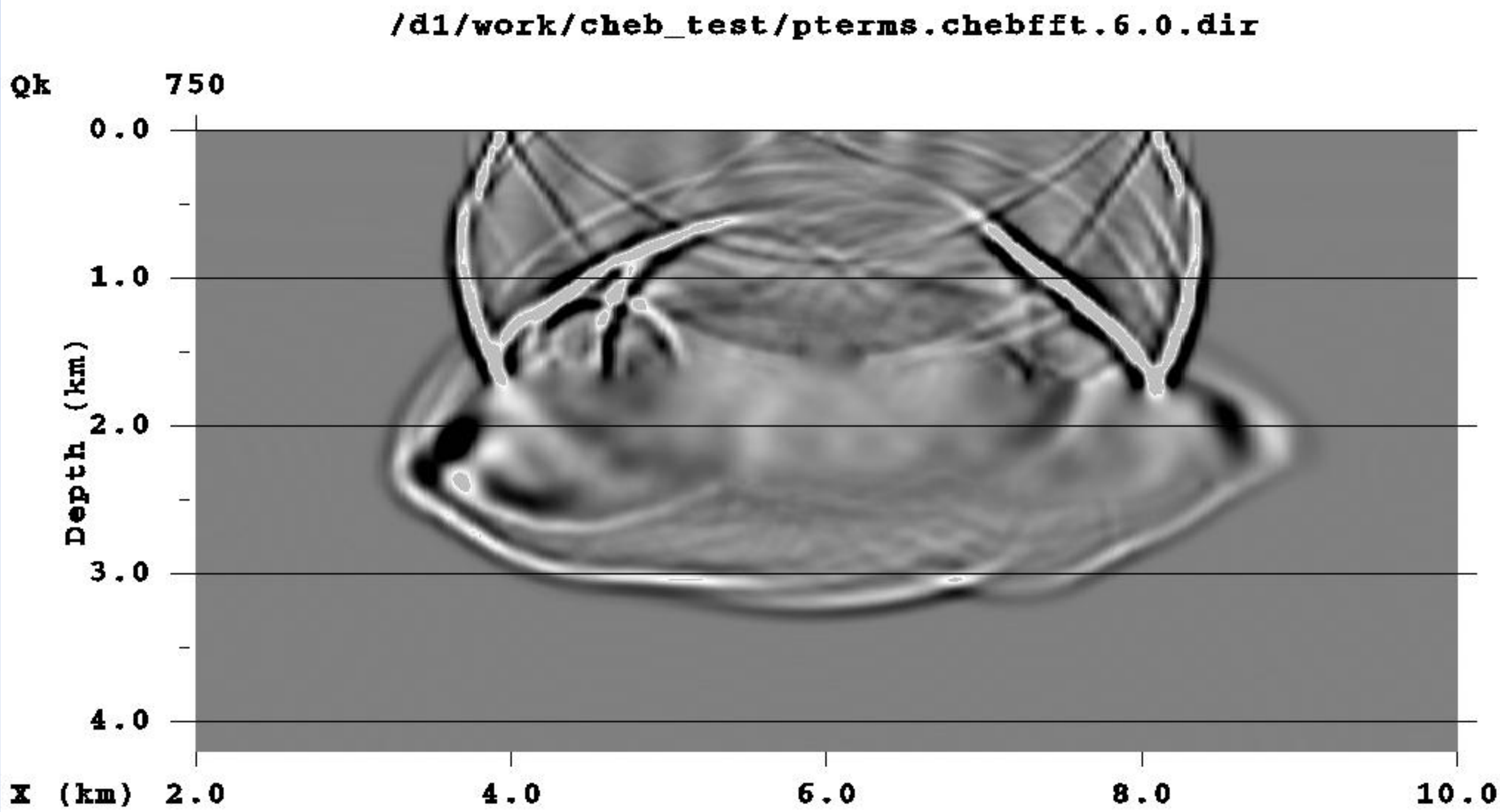
/d1/work/cheb_test/pterms.chebfft.6.0.dir
Qk 550
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0


/d1/work/cheb_test/pterms.chebfft.6.0.dir
Qk 600
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0

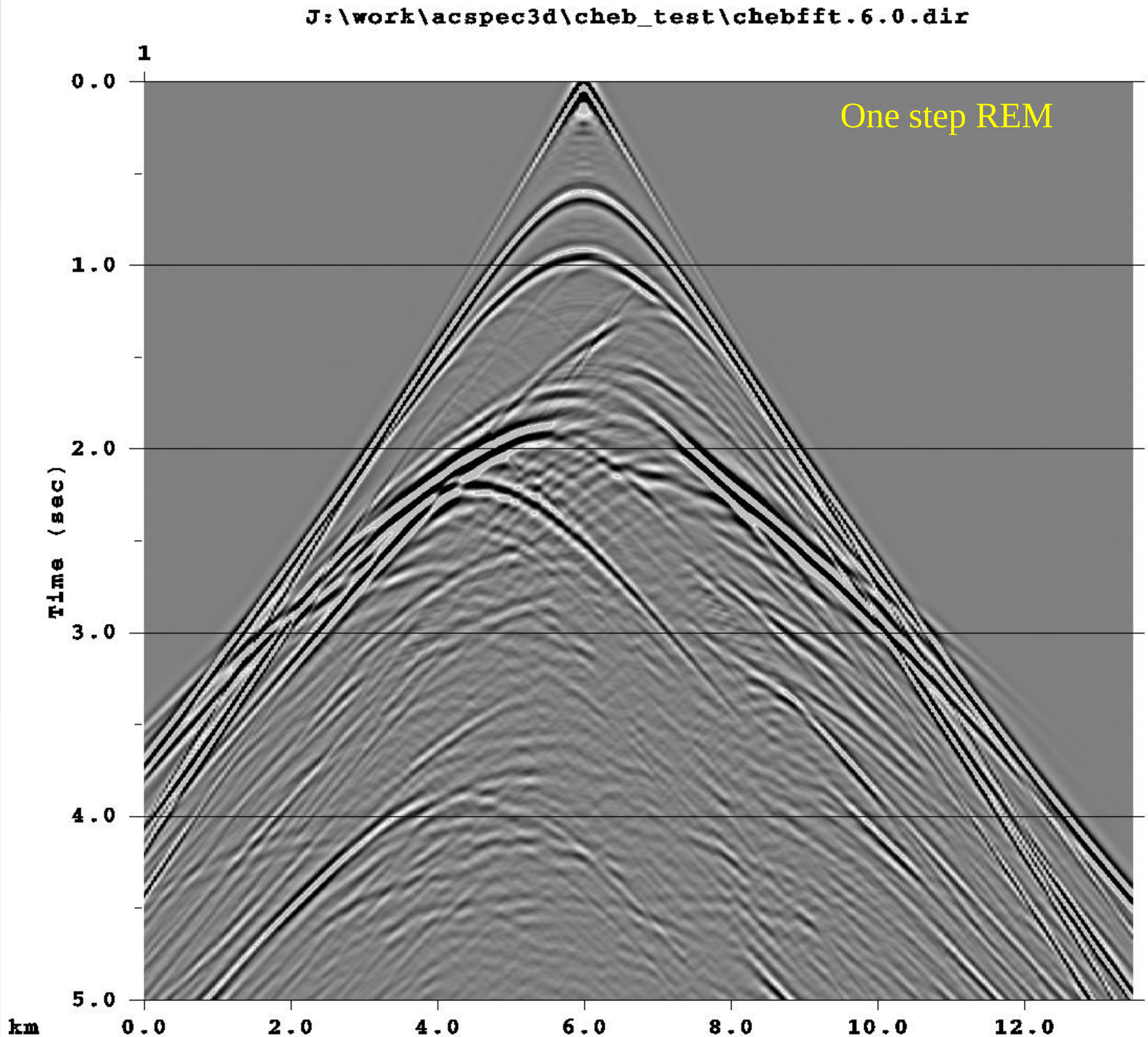


/d1/work/cheb_test/pterms.chebfft.6.0.dir
Qk 650
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0


/d1/work/cheb_test/pterms.chebfft.6.0.dir
Qk 700
0.0
1.0
2.0
3.0
4.0
Depth (km)
X (km) 2.0
4.0
6.0
8.0
10.0



We know where the waves will go but not when they will arrive !


J:\work\acspec3d\cheb_test\chebfft.6.0.dir
One step REM
Time (sec)
km
0.0
1.0
2.0
3.0
4.0
5.0
0.0
2.0
4.0
6.0
8.0
10.0
12.0

For RTM we use Recursive REM

$$p(t+\Delta t) + p(t-\Delta t) = 2\cos(L\Delta t)\,p(t)$$

for all times of interest

Recursive initial Condition: ***p0(x, y, z, t)=p(x, y, z, t)***

$$Q_0(w) = 1 \qquad \textit{and} \qquad Q_2(w) = 1 + 2w^2.$$

Calculate Chebyshev Polynomials:

apply Laplace Operator

$$-L^2 = c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$$

get next Chebyshev polynomial using recursion

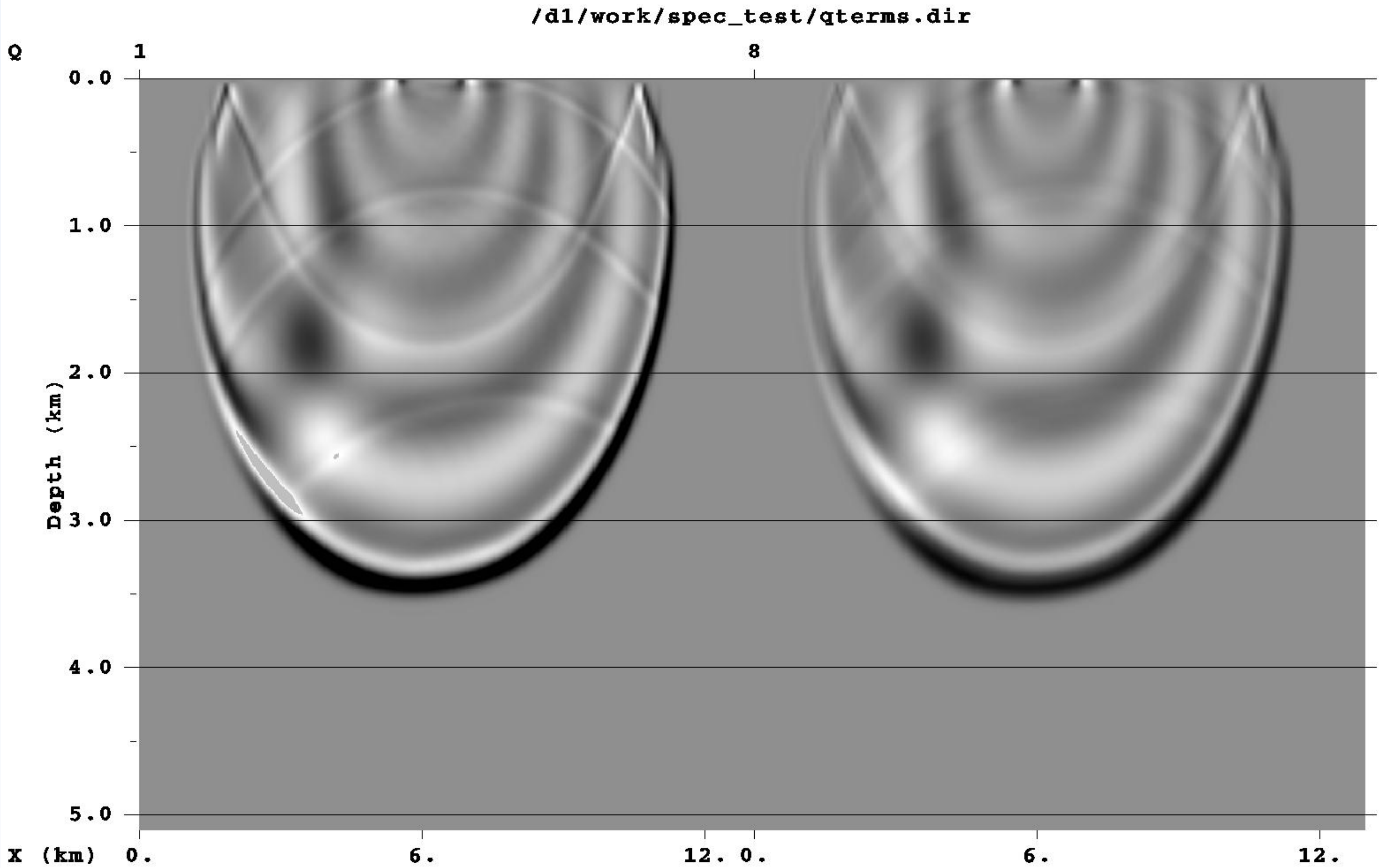
for k = 2,4, …. M = dtR

$$Q_{k+2}(w) = (4w^2+2)\,Q_k(w) - Q_{k-2}(w).$$

Integrate Chebyshev Polynomials with $J_k(dtR)$
save receiver data this time ***p(x, y, z, t)*** or
save snap shot data this time or
save both

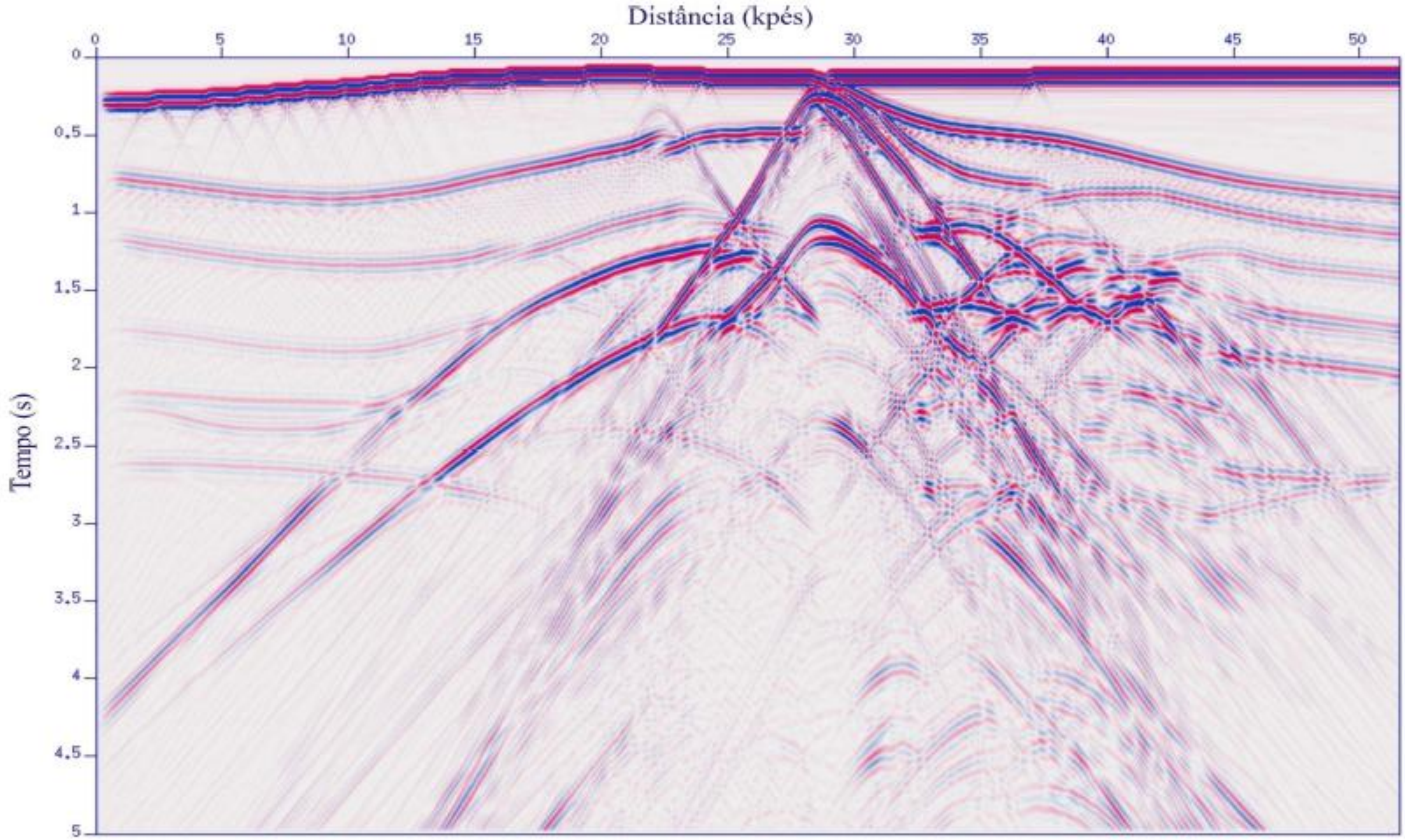
$$\sum_{k(even)}^{\infty} C_k\, J_k(\Delta t R)\, Q_k\left(\frac{iL}{R}\right)$$

Start again using ***p(x, y, z, t)*** as the initial condition



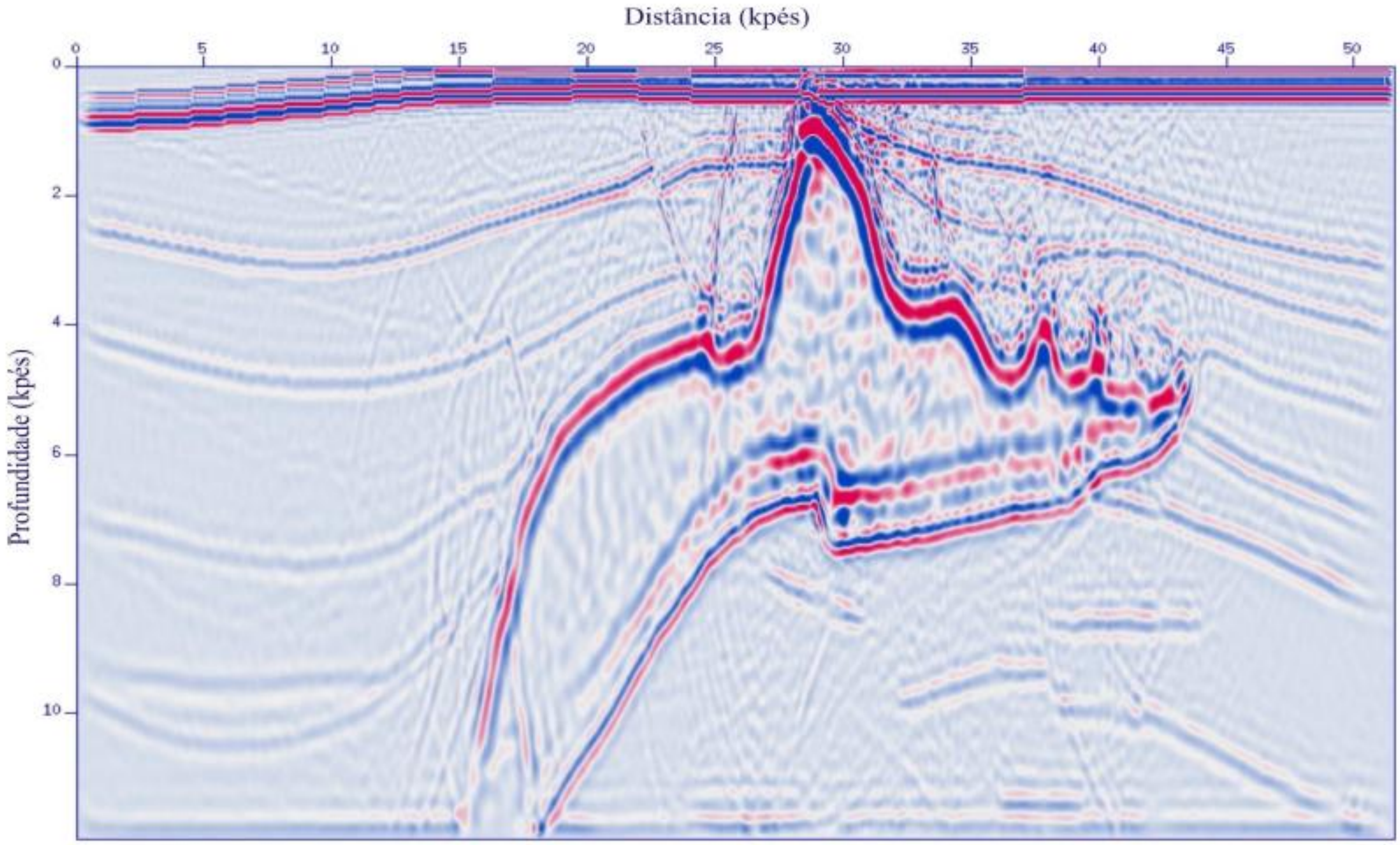
Q terms 1 and 8 for the recursive REM at the time step of 1.6 s

SEG-EAGE salt model - Zero offset section (8 ms)



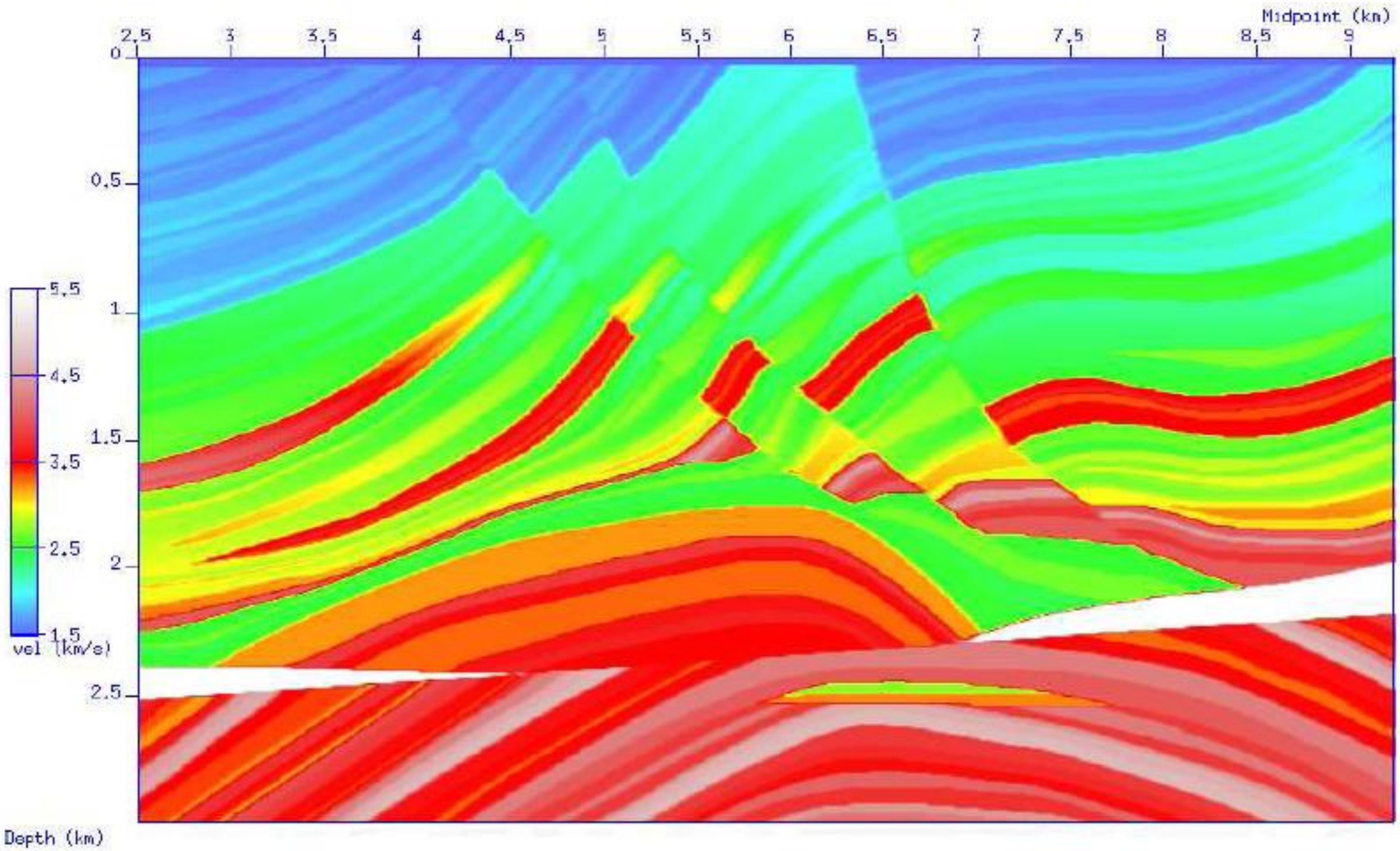
REM results for the SEG-EAGE salt model

Original data with 8ms using 8 terms



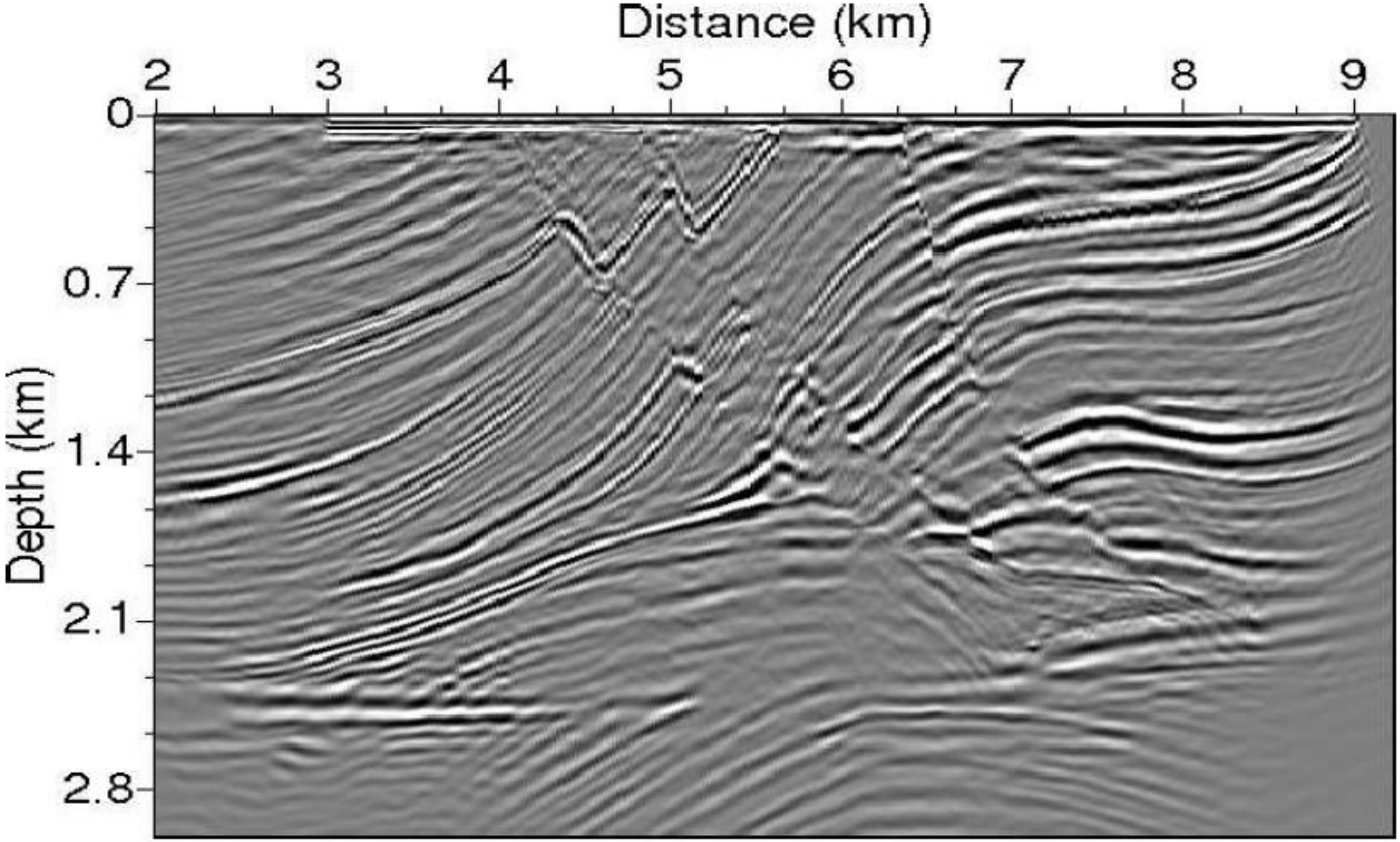
Marmoussi data - RTM pre-stack results
Parallel code - 248 nodes

Velocity field

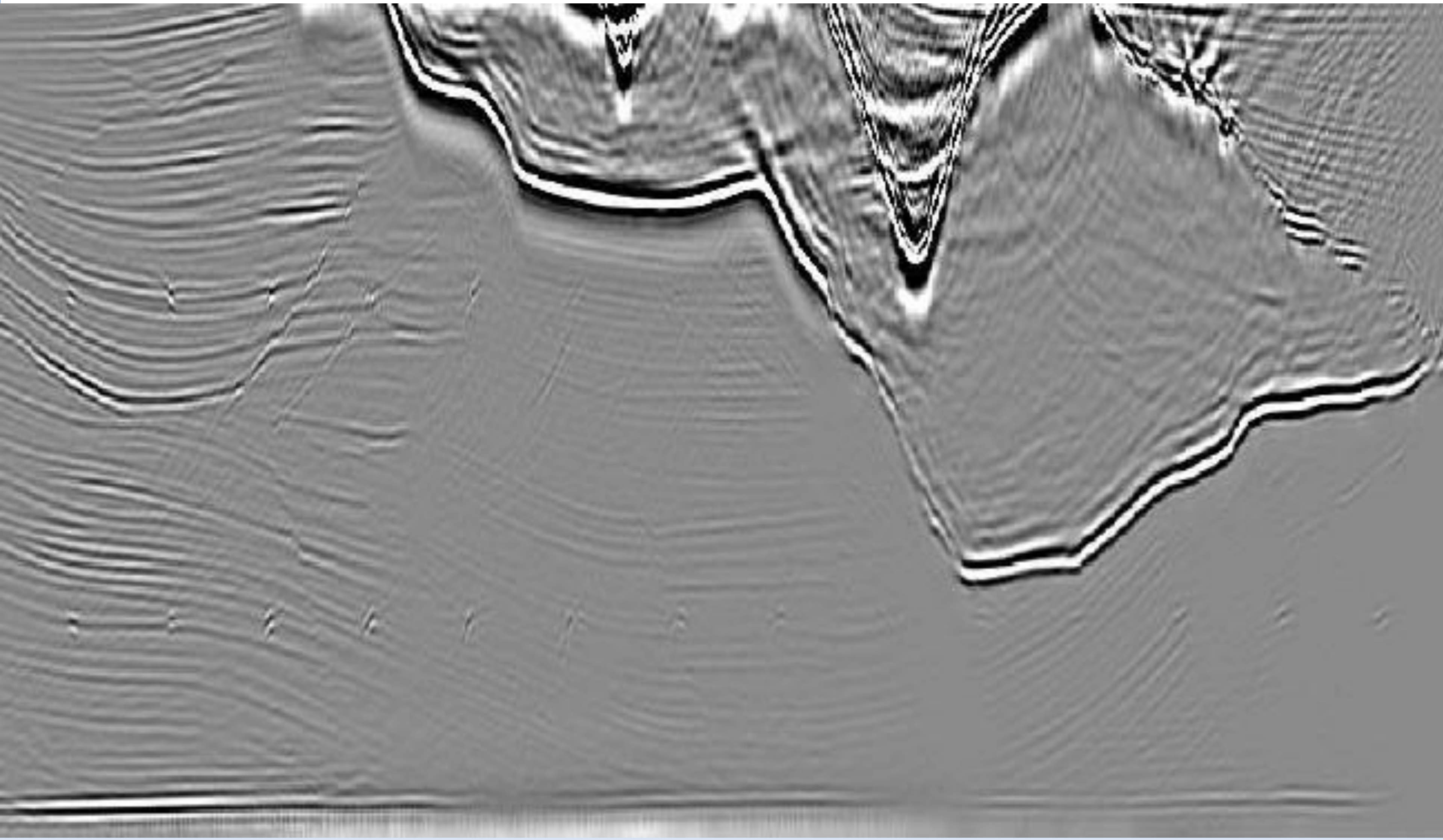


Marmoussi data - RTM pre-stack results
Parallel code - 248 nodes

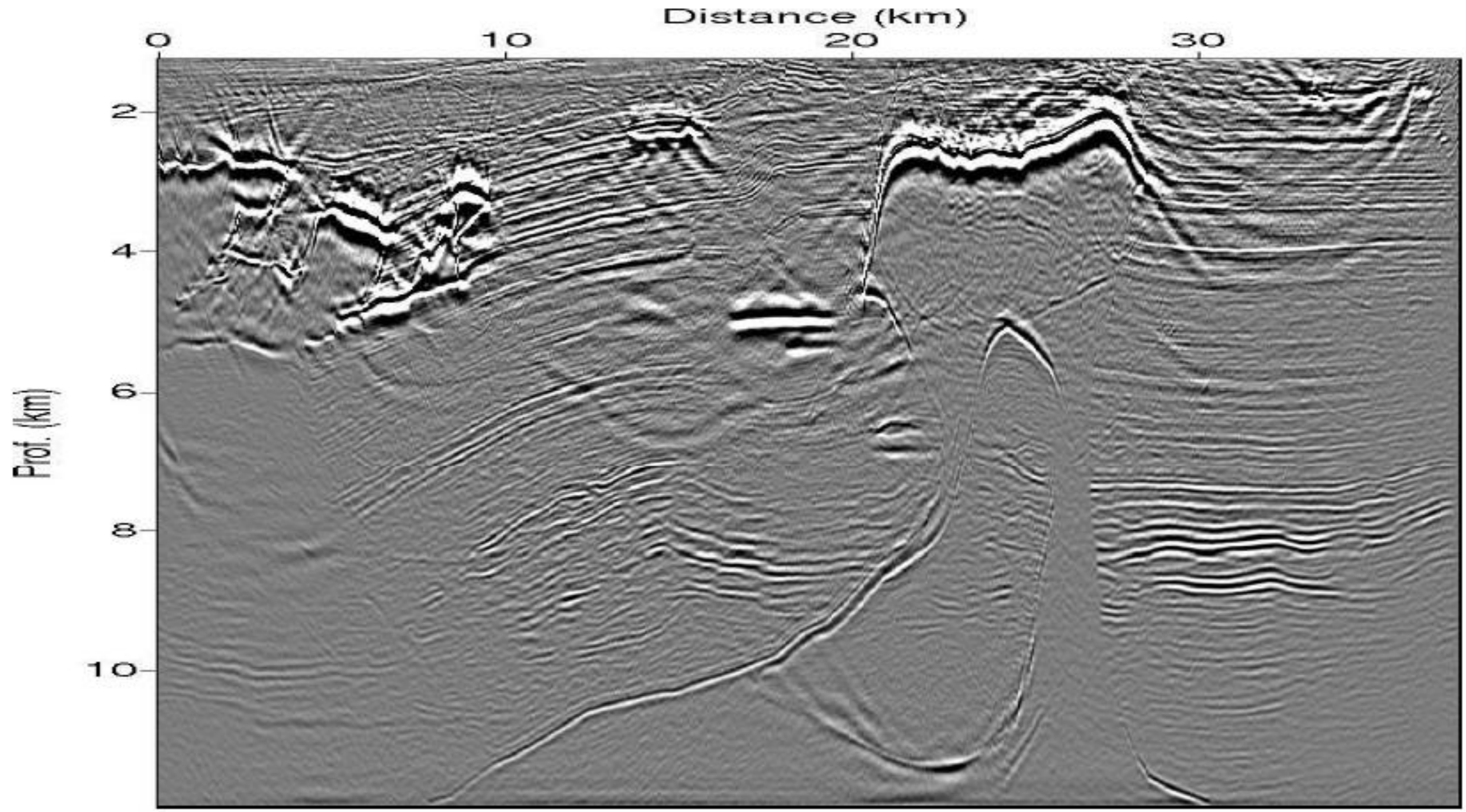
REM with original data, compute time = 2m15s ($\Delta t = 4ms$)

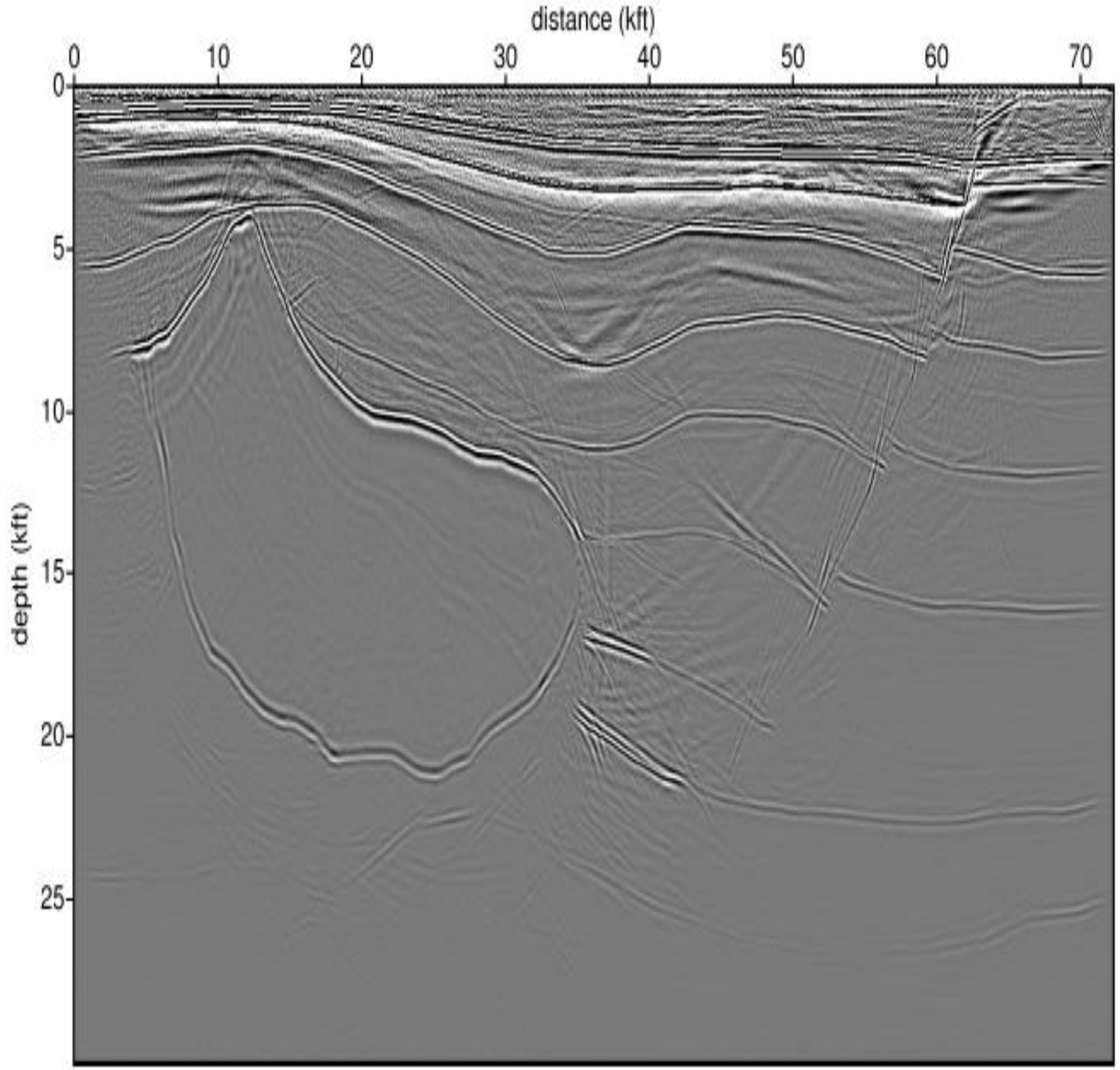


Sigsbee2a - RTM pre-stack result ($\Delta t = 8ms$)

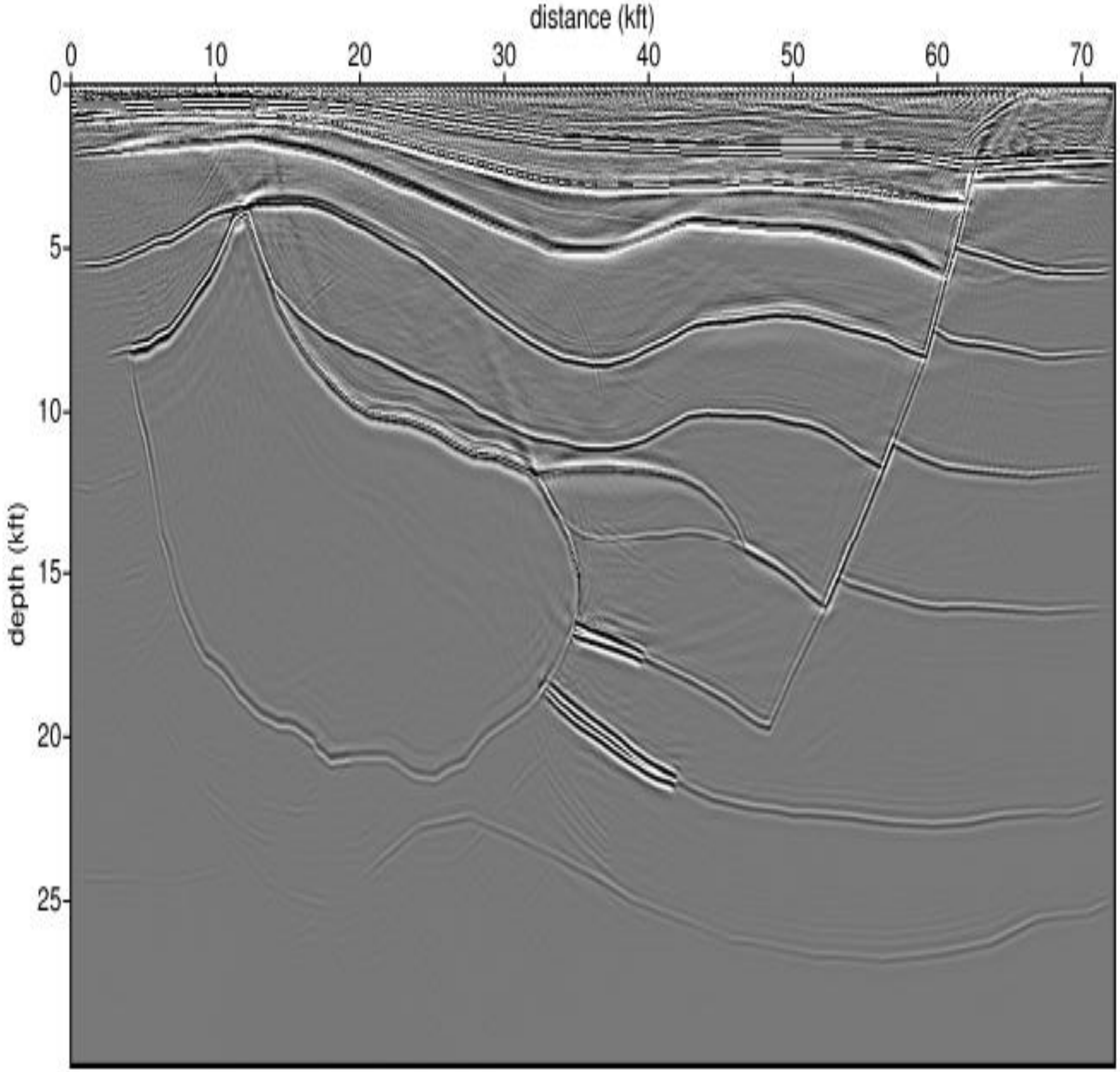


BP2004 - RTM pre-stack result ($\Delta t = 12ms$)





isotropic pre stack depth migration



anisotropic VTI pre stack depth migration

Conclusions

Time stepping:

One step REM allows the wave field at any time and in any time order to be extracted from the Chebyshev polynomials

Only the grid positions of interest (e..g receiver locations) in the Chebyshev polynomials need to be saved greatly reducing the storage requirements for 3D problems

Only a subset of the Chebyshev polynomials are actually required in the integration to produce the wave field at a specific time

One step REM and recursive REM give *nearly* identical results

Advancing in time based on recursive REM or time finite difference operators give a *similar* result if the internal *dt* for the time finite differencing is small enough to guarantee stability

Conclusions, continued

Laplacian:

4th order finite difference operators result in dispersion errors and are not suitable for high resolution simulations or RTM, increasing the order improves the result but we reach the point of diminishing returns quickly, eg 6th or 8th order

FIR or optimized finite difference operators perform much better than the standard truncated finite difference operators as they recover a larger part of the wave number band without error and have a finite spatial response. A 13 point FIR operator performs very well in the examples tested and there is marginal difference in the computation time compared with the standard finite difference operators.

Pseudo spectral operators using the FFT provide the most accurate spatial response possible (***no grid dispersion***) as no approximations to the spatial operators are required. Speed is an issue and is hardware dependent.